

## 5.3 3-folds with $b_2 \geq 3$

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REMARK 14. The Hessian of a binary form  $F \in S^3 H^\vee$  is identically zero iff  $F$  is degenerate; it is negative semi-definite if  $F$  is non-degenerate and  $\Delta(F) \leq 0$ ; it is indefinite iff  $\Delta(F) > 0$  [Ca]. Only in the indefinite case  $\Delta(F) > 0$  can the closure  $\overline{\mathcal{H}}_F := \{h \in H_{\mathbf{R}} \mid \det F'(h) \leq 0\}$  of the Hesse cone be a proper subset of  $H_{\mathbf{R}}$ .

EXAMPLE 16. Let  $P = \mathbf{P}_{\mathbf{P}^2}(E)$  be the projectivization of a rank-2 vector bundle  $E$  with Chern classes  $c_i = c_i(E)$ . The cup-form of  $P$  yields the cubic polynomial  $f = (c_1^2 - c_2)X^2 + 3(-c_1)X^2Y + 3XY^2$  whose Hessian is  $H_f = (-c_2)X^2 + c_1XY - Y^2$ . Rewriting  $H_f$  as  $H_f = -\frac{1}{4}[(2Y - c_1X)^2 + X^2(4c_2 - c_1^2)] = \frac{-1}{4}[(2Y - c_1X)^2 - \Delta(f)X^2]$  we find 3 possibilities for the Hesse cone:

- i)  $\Delta(f) < 0$ :  $\mathcal{H}_f = H^2(P, \mathbf{R}) \setminus \{0\}$
- ii)  $\Delta(f) = 0$ :  $\mathcal{H}_f = H^2(P, \mathbf{R}) \setminus L_{c_1}$  for a real line  $L_{c_1}$  depending on  $c_1$  ( $L_{c_1} = \mathbf{R}(2, c_1)$  in the coordinates  $X, Y$ )
- iii)  $\Delta(f) > 0$ :  $\mathcal{H}_f$  is an open cone whose angle is determined by  $\Delta(f) ((Z + \sqrt{\Delta(f)}X)(Z - \sqrt{\Delta(f)}X) > 0$  in coordinates  $X, Z := 2Y - c_1X$ .

### 5.3 3-FOLDS WITH $b_2 \geq 3$

Let  $X$  be a 1-connected, compact complex 3-fold with  $H_2(X, \mathbf{Z}) \cong \mathbf{Z}^{\oplus 3}$ . The cup-form of  $X$  gives rise to a curve  $C_X$  of degree 3 in the projective plane  $\mathbf{P}(H^2(X, \mathbf{C}))$ :

$$C_X := \{ \langle h \rangle \in \mathbf{P}(H^2(X, \mathbf{C})) \mid h^3 = 0 \}.$$

A first natural question is which types of plane cubic curves occur in this way?

Recall that there are 10 types of plane cubics, namely: 1) non-singular cubics, 2) irreducible cubics with a node, 3) irreducible cubics with a cusp, 4) reducible cubics consisting of a smooth conic and a transversal line, 5) smooth conics with a tangent line, 6) three lines forming a triangle, 7) three distinct lines through a common point, 8) a double line with a third skew line, 9) a triple line, 10) the trivial 'cubic' with equation 0.

LEMMA 4. *If the 3-fold  $X$  has a non-trivial Hodge number  $h^{2,0}(X) \neq 0$ , then  $C_X$  is of type 4), 6) 9) or 10).*

*Proof.* Choose basis vectors  $e^{k,l} \in H^{k,l}(X)$ , so that every  $h \in H^2(X, \mathbf{C})$  can be uniquely written as  $h = xe^{2,0} + ye^{1,1} + ze^{0,2}$ .

Then clearly  $h^3 = y[y^2(e^{1,1})^3 + 6xz(e^{2,0} \cdot e^{1,1} \cdot e^{0,2})]$ .

We now realize the cubics of types 7)-10). These cubics are degenerate, i.e. they are cones, and therefore their Hessians vanish identically. From section 4.3 we know that they can not be realized by Kählerian 3-folds.

**PROPOSITION 20.** *The plane cubics of types 7)-10) can all be realized by 1-connected, non-Kählerian 3-folds.*

*Proof.* ‘Cubics’ of type 10) can be realized by elliptic fibre bundles over surfaces  $Y$  with  $b_2(Y) = 5$ . In order to realize cubics of type 9) or 7) one blows up one or two points in an elliptic fibre bundle over a surface with  $b_2 = 4$  or 3 respectively. The realization of a type 8) cubic is a little trickier: One starts with an elliptic fibre bundle over a surface  $Y$  with  $b_2(Y) = 3$ , and blows up one of its fibers. The resulting 3-fold  $X'$  has  $b_2(X') = 2$  and  $F_{X'} \equiv 0$ . Now choose a line  $l$  in the exceptional divisor  $E$  of  $X'$ , and let  $X$  be the blow-up of  $X'$  along  $l$ . The cup-form of  $X$  yields the cubic polynomial  $x^2[y(-3l \cdot E) - x(\deg N_{C/X'})]$  with a non-zero coefficient  $-3l \cdot E = 3$ .

There are four types of complex cubics which we have been able to realize by projective 3-folds.

**PROPOSITION 21.** *Cubics of type 1), 3), 4) and 6) are realizable by 1-connected projective 3-folds.*

*Proof.* Type 1) occurs for blow-ups of complete intersections in two distinct points. The product  $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$  realizes a triangle, whereas most projective bundles over a surface with  $b_2 = 2$  lead to the union of a smooth conic and a transversal line.

Irreducible cubics with a cusp can be obtained by blowing-up a line and a point in  $\mathbf{P}^3$ . The resulting 3-fold yields the cubic polynomial  $X^3 - 3XY^2 - 2Y^3 + Z^3 = (X + Y)^2(X - 2Y) + Z^3$ .

The remaining two types of cubics are cubics with a node (type 2)), and smooth conics with a tangent line (type 5)). We do not know if these types are realizable by projective 3-folds. A non-Kählerian 3-fold whose cup-form yields a nodal cubic can be constructed: one just takes the blow-up of two suitable curves in Oguiso’s Calabi-Yau 3-fold with  $b_2 = 1$  and vanishing cup-form.

Finally we like to show that the non-emptiness condition on the index cone of a projective 3-fold with  $h^{0,2} = 0$  gives non-trivial restrictions for the possible cup-forms if  $b_2 \geq 4$ . Further investigations of this condition will appear elsewhere [Sch].

EXAMPLE 17. Let  $H$  be a free  $\mathbf{Z}$ -module of rank 4 with basis  $(e_i)_{i=1,\dots,4}$ . Consider a trilinear form  $F \in S^3 H^\vee$  and its adjoint map  $F^t: H \rightarrow S^2 H^\vee$ . The image  $F^t(h)$  of an element  $h \in H$  is in terms of the chosen basis  $(e_i)_{i=1,\dots,4}$  represented by the symmetric  $4 \times 4$ -matrix  $[[he_i e_j]]_{i,j=1,\dots,4}$ . Suppose this matrix is a diagonal sum  $[[he_i e_j]]_{i,j=1,2} \oplus [[he_k e_l]]_{k,l=3,4}$  such that the determinants of both  $2 \times 2$ -matrices are negative for every  $h \in H \setminus \{0\}$ .

In this case  $F^t(h)$  were of signature  $(1, -1, 1, -1)$  for every  $h \in H \setminus \{0\}$ , and we would have  $I_F = \mathcal{H}_F = \emptyset$ .

All these conditions can be met, e.g. by setting  $e_1^2 e_2 = e_2^3 = e_3^2 e_4 = e_4^3 = 1$ ,  $e_1 e_2^2 = e_3 e_4^2 = 2$ , and  $e_i e_j e_k = 0$  otherwise. In this particular case the image of  $h = \sum_{i=1}^4 h_i e_i$  under  $F^t$  is represented by the matrix

$$\left[ \begin{array}{cc|cc} h_2 & h_1 + 2h_2 & & \\ h_1 + 2h_2 & 2h_1 + h_2 & & \\ \hline & & h_4 & h_3 + 2h_4 \\ & 0 & h_3 + 2h_4 & 2h_3 + h_4 \end{array} \right],$$

which has a positive determinant unless  $h = 0$ .

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