8. OUTER AUTOMORPHISMS OF GROUPS OF TYPE F

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$$\chi_1(G)(ht^{\nu q}) = \left(\sum_{n \geqslant 0} \sum_{i=0}^{\nu q - 1} (-1)^n A(\mathrm{trace}([\tilde{f}_n][f_n^i])), \; -(q/r)\nu \sum_{i=0}^{r-1} L(f^i) \right)$$

and

$$\chi_1(G; \mathbf{Q})(ht^{\nu q}) = \left(0, \; -(q/r)\nu \sum_{i=0}^{r-1} L(f^i) \right) = (q/r)\nu \sum_{i=0}^{r-1} L(f^i)\{t\}$$

where $h \in \mathrm{Fix}(\theta) \cap h_0^{-\nu q/r} Z(H)$. □

Similarly, one can read off formulae for $\tilde{X}_1(G)$ from Theorem 6.14 and the rational version from Theorem 6.16.

8. Outer automorphisms of groups of type $\mathscr{F}$

In this section we apply the preceding theory to prove the following theorem which relates the algebraic topology of an automorphism $\theta : H \to H$ of a group H of type $\mathscr{F}$ such that θ has finite order in $\mathrm{Out}(H)$ to the fixed group of θ.

THEOREM 8.1. *Let* H *be a group of type* $\mathscr{F}$ *which has the Weak Bass Property over* $\mathbf{Q}$. *Suppose that* $\theta : H \to H$ *is an automorphism whose order in* $\mathrm{Out}(H)$ *is* $r \geqslant 1$. *If the sum of the Lefschetz numbers* $\sum_{i=0}^{r-1} L(\theta^i)$ *is non-zero then* $Z(H) \cap \mathrm{Fix}(\theta) = (1)$.

Before proving this we note that the quantity $\sum_{i=0}^{r-1} L(\theta^i)$ appearing above has the following interpretation:

PROPOSITION 8.2. $\sum_{i=0}^{r-1} L(\theta^i)$ *is* r *times the Euler characteristic of the* θ*-invariant part of the homology of* H, *i.e.,*

$$\sum_{i=0}^{r-1} L(\theta^i) = r \sum_{j \geqslant 0} (-1)^j \,\mathrm{rank}\,\ker(\mathrm{id} - \theta_j : H_j(H) \to H_j(H)) \,.$$

Proof. By elementary linear algebra, for any square complex matrix A with $A^r = I$ we have $\mathrm{trace}(\sum_{i=0}^{r-1} A^i) = r \dim \ker(I - A)$. The conclusion easily follows. □

Proof of Theorem 8.1. Let G be the semidirect product $G = H \times_\theta T$ where T is infinite cyclic. By Lemma 8.7, below, G also has the WBP over $\mathbf{Q}$. Applying Theorem 7.11 to G, we have that $\chi_1(G; \mathbf{Q}) \neq 0$. By

Theorem 5.4, $Z(G)$ is infinite cyclic. By Corollary 7.9 there is an exact sequence $1 \to Z(H) \cap \mathrm{Fix}(\theta) \to Z(G) \xrightarrow{p_*} q\mathbf{Z} \to 1$ where the period of θ, q, is positive. It follows that $Z(H) \cap \mathrm{Fix}(\theta) = (1)$. □

If $\chi(H) \neq 0$ then $Z(H) = (1)$ by Proposition 2.4 and consequently $Z(H) \cap \mathrm{Fix}(\theta) = (1)$ in this case. If $\chi(H) = L(\theta^0) = 0$ then $\sum_{i=0}^{r-1} L(\theta^i) = \sum_{i=1}^{r-1} L(\theta^i)$. These observations yield the following corollaries of Theorem 8.1:

COROLLARY 8.3. *Let H be a group of type $\mathscr{F}$ which has the WBP over* $\mathbf{Q}$. *Suppose that $\theta: H \to H$ is an automorphism of order* 2 *in* $\mathrm{Out}(H)$. *If $L(\theta) \neq 0$ then* $Z(H) \cap \mathrm{Fix}(\theta) = (1)$. □

COROLLARY 8.4. *Let H be a group of type $\mathscr{F}$ which has the WBP over* $\mathbf{Q}$. *Suppose $Z(H) \neq (1)$, the automorphism $\theta: H \to H$ has finite order r in* $\mathrm{Out}(H)$ *and the restriction of θ to $Z(H)$ is the identity. Then* $\sum_{i=1}^{r-1} L(\theta^i) = 0$.

Proof. Since the restriction of θ to $Z(H)$ is the identity, $Z(H) \cap \mathrm{Fix}(\theta) = Z(H) \neq (1)$. □

An automorphism which has finite order in $\mathrm{Out}(H)$ may have infinite order in $\mathrm{Aut}(H)$. If θ has finite order in $\mathrm{Aut}(H)$, the Weak Bass Property hypothesis can be dispensed with in Theorem 8.1 and Corollary 8.3:

PROPOSITION 8.5. *Let H be a group of type $\mathscr{F}$. Suppose that $\theta: H \to H$ has finite order in* $\mathrm{Aut}(H)$ *and $L(\theta) \neq 0$. Then* $Z(H) \cap \mathrm{Fix}(\theta) = (1)$.

Proof. Let $\omega \in Z(H) \cap \mathrm{Fix}(\theta)$. We use the terminology of [Br]. Let Z be a finite $K(H, 1)$. Choose an essential fixed point, v, of $f: Z \to Z$ (inducing θ) as the basepoint of Z. There is a homotopy $K: f \simeq f$ such that $K(v, \cdot)$ represents ω. The fixed point v is K-related to some fixed point u of f [Br, p. 92]. Hence, for some $s > 0$, v is J-related to v, where J is the s-fold concatenation $K \star \cdots \star K$. Then there exists $\sigma \in H$ such that $\omega^s = \sigma\theta(\sigma^{-1})$; compare [G]. As in the proof of Proposition 7.7, we get $\omega^{rs} = \prod_{i=0}^{r-1} \theta^i(\sigma\theta(\sigma^{-1})) = 1$, so $\omega = 1$. □

Note that $\sum_{i=1}^{r-1} L(\theta^i) \neq 0$ implies one of the $L(\theta^i)$'s is non-zero. Since $\mathrm{Fix}(\theta) \subset \mathrm{Fix}(\theta^i)$ for $i \geqslant 0$, we recover Theorem 8.1 (but without the Bass Conjecture hypothesis) in the special case where θ has finite order in $\mathrm{Aut}(H)$.

The remainder of this section is devoted to the proof of Lemma 8.7 used above.

LEMMA 8.6. *Suppose that the group* H *has the WBP over* $\mathbf{Q}$. *Let* T *be an infinite cyclic group. Then the product group* $H \times T$ *also has the WBP over* $\mathbf{Q}$.

Proof. Let $G = H \times T$. Identify H with $H \times \{1\} \subset G$. We use the notation of § 5. By Schafer's theorem [Sch, p. 224] applied to the normal subgroup $H \subset G$, the image of $T_0: K_0(\mathbf{Q}G) \to HH_0(\mathbf{Q}G)$ lies in $HH_0(\mathbf{Q}G)_H$. Let $p: G \to H$ be the projection homomorphism. There is a commutative diagram:

$$\begin{array}{ccccc} K_0(\mathbf{Q}G) & \stackrel{T_0}{\to} & HH_0(\mathbf{Q}G)_H & \stackrel{\varepsilon_*}{\to} & \mathbf{Q} \\ p_* \downarrow & & p_* \downarrow & & \| \\ K_0(\mathbf{Q}H) & \stackrel{T_0}{\to} & HH_0(\mathbf{Q}H) & \stackrel{\varepsilon_*}{\to} & \mathbf{Q} \end{array}$$

Write $HH_0(\mathbf{Q}G)_H = HH_0(\mathbf{Q}G)_{C(1)} \oplus HH_0(\mathbf{Q}G)''_H$ where $HH_0(\mathbf{Q}G)''_H$ is the direct sum of the $HH_0(\mathbf{Q}G)_{C(g)}$'s over $C(g) \in c(H) - \{C(1)\}$; also, $HH_0(\mathbf{Q}H) = HH_0(\mathbf{Q}H)_{C(1)} \oplus HH_0(\mathbf{Q}H)'$. By hypothesis, H has the WBP over $\mathbf{Q}$, i.e. the composite

$$K_0(\mathbf{Q}H) \stackrel{T_0}{\to} HH_0(\mathbf{Q}H) \to HH_0(\mathbf{Q}H)' \stackrel{\varepsilon_*}{\to} \mathbf{Q}$$

is zero. Since $p_*(HH_0(\mathbf{Q}G)_{C(1)}) \subset HH_0(\mathbf{Q}H)_{C(1)}$ and $p_*(HH_0(\mathbf{Q}G)''_H) \subset HH_0(\mathbf{Q}H)'$, the conclusion follows. □

LEMMA 8.7. *Suppose that the group* H *has the WBP over* $\mathbf{Q}$ *and that* $\theta: H \to H$ *is an automorphism whose image in the group of outer automorphisms of* H *has finite order. Then the semidirect product* $H \times_\theta T$ *also has the WBP over* $\mathbf{Q}$.

Proof. Let $G = H \times_\theta T \equiv \langle H, t \mid tht^{-1} = \theta(h) \text{ for } h \in H \rangle$. Let n be the order of θ in the group outer automorphisms of H. Then the subgroup G' of G generated by H and t^n is isomorphic to $H \times T$; furthermore, G' is normal and of finite index, n, in G. There is a "transfer" homomorphism trans: $HH_0(\mathbf{Q}G) \to HH_0(\mathbf{Q}G')$ defined as follows. Given $g \in G$, we can write $gt^i = t^{\sigma(i)} g_i$ for $i = 0, \ldots, n-1$ where $g_i \in G'$ and σ is a permutation of $\{0, \ldots, n-1\}$. Let $\mathrm{Fix}(\sigma) = \{i \mid \sigma(i) = i\}$. Then $\mathrm{trans}(C(g)) = \sum_{i \in \mathrm{Fix}(\sigma)} C(g_i)$. Observe that if $g \in G'$ then $\mathrm{Fix}(\sigma) = \{0, \ldots, n-1\}$

because G' is normal in G. In particular, $\varepsilon_*(\mathrm{trans}(C(g))) = n$ if $g \in G'$. There is a commutative diagram:

$$\begin{array}{ccc} K_0(\mathbf{Q}G) & \overset{T_0}{\rightarrow} & HH_0(\mathbf{Q}G) \\ \text{res}\downarrow & & \text{trans}\downarrow \\ K_0(\mathbf{Q}G') & \overset{T_0}{\rightarrow} & HH_0(\mathbf{Q}G') \end{array}$$

where $\mathrm{res}: K_0(\mathbf{Q}G) \rightarrow K_0(\mathbf{Q}G')$ is obtained by regarding a projective $\mathbf{Q}G$ module as a projective $\mathbf{Q}G'$ module; see [Bass] for details concerning the finite index transfer.

Recall that $HH_0(\mathbf{Q}G) = HH_0(\mathbf{Q}G)_H \oplus HH_0(\mathbf{Q}G)'_H$ where $HH_0(\mathbf{Q}G)'_H$ is the direct sum of the summands $HH_0(\mathbf{Q}G)_{C(g)}$ corresponding to the conjugacy classes not represented by elements of H. By Schafer's theorem [Sch, p. 224] applied to the normal subgroup $H \subset G$, the image of $T_0: K_0(\mathbf{Q}G) \rightarrow HH_0(\mathbf{Q}G)$ lies in $HH_0(\mathbf{Q}G)_H$. Thus we can replace $HH_0(\mathbf{Q}G)$ with $HH_0(\mathbf{Q}G)_H$ in the above diagram and obtain the commutative diagram:

$$\begin{array}{ccccc} K_0(\mathbf{Q}G) & \overset{T_0}{\rightarrow} & HH_0(\mathbf{Q}G)_H & \overset{\varepsilon_*}{\rightarrow} & \mathbf{Q} \\ \text{res}\downarrow & & \text{trans}\downarrow & & \times n\downarrow \\ K_0(\mathbf{Q}G') & \overset{T_0}{\rightarrow} & HH_0(\mathbf{Q}G') & \overset{\varepsilon_*}{\rightarrow} & \mathbf{Q} \end{array}$$

(the right square commutes because $H \subset G'$ and because of the observation made above). Write $HH_0(\mathbf{Q}G)_H = HH_0(\mathbf{Q}G)_{C(1)} \oplus HH_0(\mathbf{Q}G)''_H$ where $HH_0(\mathbf{Q}G)''_H$ is the direct sum of the $HH_0(\mathbf{Q}G)_{C(g)}$'s over $C(g) \in c(H) - \{C(1)\}$; also, $HH_0(\mathbf{Q}G') = HH_0(\mathbf{Q}G')_{C(1)} \oplus HH_0(\mathbf{Q}G')'$. Then $\mathrm{trans}(HH_0(\mathbf{Q}G)_{C(1)}) \subset HH_0(\mathbf{Q}G')_{C(1)}$ and $\mathrm{trans}(HH_0(\mathbf{Q}G)''_H) \subset HH_0(\mathbf{Q}G')'$. By Lemma 8.6, G' has the WBP over $\mathbf{Q}$, i.e. the composite $K_0(\mathbf{Q}G') \overset{T_0}{\rightarrow} HH_0(\mathbf{Q}G') \rightarrow HH_0(\mathbf{Q}G')' \overset{\varepsilon_*}{\rightarrow} \mathbf{Q}$ is zero. The conclusion follows from the above diagram. □

9. TRACE FORMULAE FOR HOMOLOGICAL INTERSECTIONS

The goal of this section is to prove a "trace formula" (Theorem 9.13) for the homological intersection of the graph of a map $F: M \times Y \rightarrow M$ with the graph of the projection map $p: M \times Y \rightarrow M$ where Y is a closed oriented manifold and M is a compact oriented manifold. This result will be applied in § 10 to complete the proof of Theorem 1.1.