HIGHER EULER CHARACTERISTICS (I)

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To Peter Hilton on the occasion of his 70-th birthday.

ABSTRACT. The classical Euler characteristic $\chi \equiv \chi_0$ of a finite complex lies at the bottom of a sequence of homotopy invariants. The next invariant in this sequence χ_1 is introduced here and studied in some detail. The rest of the sequence, χ_n with $n \geqslant 2$, will be discussed in a sequel paper. Applications to geometric group theory are found by considering the behavior of χ_1 on an aspherical finite complex of fundamental group G. Just as the $\chi(G) \neq 0$ implies that the center of G is trivial (Gottlieb's Theorem), it is shown here that (under a weak additional hypothesis and using rational coefficients) $\chi_1(G) \neq 0$ implies that the center of G is infinite cyclic. We also find a generalization of Gottlieb's Theorem in which the Lefschetz number of an automorphism of G is related to the fixed subgroup of the automorphism.

INTRODUCTION

From our point of view, the classical Euler characteristic of a finite complex is "zero-th order". In this paper we introduce a "first order" analog, a new invariant in topology and group theory. In a sequel paper and in [GNO] we extend these ideas to an "n-th order" Euler characteristic for all positive n.

For a finite complex X, the new invariant $\chi_1(X; R)$, defined in § 1, comes in different forms, depending on the coefficient ring R; and a more sophisticated version $\tilde{\chi}_1(X; R)$ defined in § 2, involves the universal cover of X. By contrast, the classical analogs of these are essentially the same, namely the integer $\chi(X)$. We should tell the reader from the start that all our first order invariants are trivial if X is simply connected.

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The paper begins with three rather different definitions of $\chi_1(X; R)$, a discussion of their equivalence, and some motivation for these definitions. Our point of view is geometric, but for readers more interested in homotopy theory we include (at the end of § 1) a brief discussion of a fourth definition in terms of stable homotopy theory.

Next, we discuss the computation of $\chi_1(X; R)$ for 1-complexes, certain 2-complexes, 3-dimensional lens spaces, circle bundles and mapping tori.

In § 5 and § 7, we apply these ideas to group theory. Motivated by Gottlieb's theorem [Got] that if X is a finite aspherical complex with fundamental group G and if $\chi(G) \equiv \chi(X) \neq 0$ then the center of G is trivial, we find an analog (Theorem 5.4) which says, roughly, that if $\chi_1(G; \mathbf{Q}) \equiv \chi_1(X; \mathbf{Q}) \neq 0$ then the center of G is infinite cyclic. This leads us to surprising generalization of Gottlieb's theorem (Theorem 8.1). In this theorem, one is given an automorphism θ of G induced by a map $f: X \to X$. By the *Lefschetz number*, $L(\theta)$, of θ we mean the Lefschetz number of f. We prove (under a weak K-theoretic hypothesis on G) that if θ has order r in the group of outer automorphisms of G and if $\sum_{i=0}^{r-1} L(\theta^i) \neq 0$ then the intersection of the center of G and the fixed subgroup of θ is trivial. We do not know of a previous theorem which relates so directly the fixed subgroup of an automorphism to the classical fixed point theory of the associated map.

We also introduce a more refined invariant $\tilde{\chi}_1(X) \in H^1(\Gamma, HH_1(\mathbf{Z}G))$ where $HH_1(\mathbf{Z}G)$ is the first Hochschild homology group of $\mathbf{Z}G$ (see § 1). This is an analog of what one obtains when one computes the classical Euler characteristic as a Hattori-Stallings trace in the universal cover of X. In the classical case one essentially recovers $\chi(X)$, but a significant distinction appears in the case of the "higher order" invariants. In a natural manner, $\tilde{\chi}_1(X)$ maps to $\chi_1(X)$ regarded as an element of $H^1(\Gamma, H_1(G))$. Applications of $\tilde{\chi}_1$ to characteristic classes and Seifert fiber spaces will be given in [GN$_5$].

The ideas presented here are an outgrowth of the one-parameter fixed point theory developed in [GN$_1$] and its application to dynamics in [GN$_2$]. A summary has appeared as [GN$_3$]. Most of this paper can be read independently of [GN$_1$] and [GN$_2$]. We make modest use of a few technical propositions from [GN$_1$] in § 2 and § 3; in § 10 a difficult result from [GN$_1$] is invoked.

One of the definitions of $\chi_1(X; R)$ employs a formula introduced more than twenty years ago in [Kn]; we thank Boris Okun for drawing our attention to that paper.

1. Three definitions of the first order Euler characteristic

Recall three definitions of the Euler characteristic, $\chi(X)$, of a finite complex X.

Definition A_0. $\chi(X) = \sum_{k \geqslant 0}(-1)^k$ (number of k-cells in X).

Definition B_0. $\chi(X; R) = \sum_{k \geqslant 0}(-1)^k \operatorname{rank}_R H_k(X; R)$ where R is a principal ideal domain. (This integer is independent of R.)

When X is an oriented manifold, M, we also have:

Definition C_0. $\chi(M) =$ intersection number of the graph of the identity map of M with itself.

We will introduce a higher analog called "the first order Euler characteristic" of X. There will be three analogous definitions, labelled A_1, B_1, and C_1 corresponding to the above definitions of the classical Euler characteristic. We prove in § 10 that under appropriate hypotheses these new definitions are equivalent.

First, we establish some notation. Let X be a finite connected CW complex with base vertex v. Write $G \equiv \pi_1(X, v)$ and $\Gamma \equiv \pi_1(X^X, \mathrm{id})$ where X^X is the function space of all continuous maps $X \to X$. Each $\gamma \in \Gamma$ can be represented by a cellular homotopy $F^\gamma : X \times I \to X$ such that $F_0^\gamma = F_1^\gamma = \mathrm{id}_X$. Orient the cells of X, thus establishing a preferred basis for the integral cellular chains $(C_*(X), \partial)$. Choose a lift, $\tilde{e}$, in the universal cover, $\tilde{X}$, for each cell e of X, and orient $\tilde{e}$ compatibly with e. Regard the cellular chain complex $(C_*(\tilde{X}), \tilde{\partial})$ as a free right $\mathbf{Z}G$-module chain complex with preferred basis $\{\tilde{e}\}$. Let $D_*^\gamma : C_*(X) \to C_{*+1}(X)$ be the chain homotopy induced by F^γ.

Sign Convention. If e is an oriented k-cell of X then $D_k(e)$ is the $(k+1)$-chain $(-1)^{k+1} F_*(e \times I) \in C_{k+1}(X)$, where $e \times I$ is given the product orientation.

Let R be a commutative ring. Regard ${}_R\tilde{\partial}_k \equiv \tilde{\partial}_k \otimes \mathrm{id} : C_k(\tilde{X}) \otimes R \to C_{k-1}(\tilde{X}) \otimes R$ and ${}_R D_k^\gamma \equiv D_k^\gamma \otimes \mathrm{id} : C_k(X) \otimes R \to C_{k+1}(X) \otimes R$ as matrices over RG and R respectively using the preferred bases. The abelianization homomorphism $A : G \to G_{ab} \cong H_1(X)$ extends to a homomorphism of R-modules $A : RG \to H_1(X; R) = H_1(X) \otimes R$.

We can now state the first definition of our *first order Euler characteristic* with coefficients in a commutative ring R. It is a homomorphism $\chi_1(X; R) : \Gamma \to H_1(X; R)$. When $R = \mathbf{Z}$ we write, in abbreviated form, $\chi_1(X) : \Gamma \to H_1(X)$. Note that Γ is abelian, and when X is aspherical, $\Gamma \cong Z(G)$, the center of G; see Proposition 1.3.

Definition A_1. Let R be a commutative ring of coefficients.

$$\chi_1(X;R)(\gamma) = \sum_{k \geqslant 0} (-1)^{k+1} A\left(\operatorname{trace}({}_R\tilde{\partial}_{k+1}\, {}_R D_k^\gamma)\right).$$

Here, we are multiplying RG-matrices by R-matrices to obtain RG-matrices. Note that $\chi_1(X;R)(\gamma) = \chi_1(X)(\gamma) \otimes 1$. We will show (Corollary 2.10) that this formula is independent of the various choices that have been made. Note that in order to know the right hand side, we must have information at the chain level, namely the matrices ${}_R\tilde{\partial}_{k+1}$ and ${}_R D_k^\gamma$. Definition A_1 is the "reduction" of a trace in 1-dimensional Hochschild homology; the corresponding trace (of the identity map) in 0-dimensional Hochschild homology "reduces" in the same way to Definition A_0; see §2 for more on this.

Our second definition requires the assumption that $H_*(X;R)$ be a free R-module where R is a principal ideal domain. This will be true, for example, if R is a field. For each $k \geqslant 0$, choose a basis $\{b_1^k, \ldots, b_{\beta_k}^k\}$ for $H_k(X;R)$. Let $\{\bar{b}_j^k\}$ be the corresponding dual basis for $H^k(X;R)$. Let $\Phi^\gamma : X \times S^1 \to X$ be the obvious quotient obtained from F^γ, above. By means of the Künneth formula, Φ^γ induces $\Phi_*^\gamma : H_k(X;R) \otimes H_1(S^1;R) \to H_{k+1}(X;R)$. Let $u \in H_1(S^1;R)$ be the generator which defines the usual orientation on S^1.

Definition B_1. Let R be a principal ideal domain. Suppose that $H_*(X;R)$ is a free R-module.

$$\chi_1(X;R)(\gamma) = \sum_{k \geqslant 0} (-1)^{k+1} \sum_j \bar{b}_j^k \cap \Phi_*^\gamma(b_j^k \otimes u)$$

where $\cap$ is the cap product in the sense of [D_2].

It is straightforward to show that the formula in Definition B_1 is independent of the choice of basis for $X_*(X;R)$.

Remark. Throughout this paper we use Dold's conventions [D_2] for cap and cup products. These conventions are the same as those of [MS] but differ from those of [Sp]. Writing $\cap'$ and $\cup'$ for the cap and cup products of [Sp], we have $x \cap y = (-1)^{|x|(|x|-|y|)} x \cap' y$ and $u \cup v = (-1)^{|u||v|} u \cup' v$ where "$|\ |$" denote the degree of a homology or cohomology class.

The above expression for $\chi_1(X;R)(\gamma)$ can also be written:

$$\chi_1(X;R)(\gamma) = \sum_{i=1}^{\beta_1} \sum_{k,j} (-1)^{k+1} \langle \bar{b}_i^1 \cup \bar{b}_j^k, \Phi_*^\gamma(b_j^k \otimes u)\rangle b_i^1$$

where $\langle \cdot , \cdot \rangle$ denotes the Kronecker pairing. A trace formula of this kind, for parametrized maps $X \times Y \to X$ was introduced by R. J. Knill in [Kn]. In order to know the right hand side in Definition B_1, we only need homological information about Φ^γ and cup product information about $H^*(X; R)$. The theory of [Kn] when applied to the identity map of X yields Definition B_0; hence the analogy with Definition B_1 (also see § 10).

Our third definition, Definition C_1 below, is an analog of the geometric Definition C_0 of $\chi(X)$. Let M be a compact oriented smooth (or PL) manifold with boundary. The *fixed point set* of F^γ is $\mathrm{Fix}(F^\gamma) \equiv \{(x, t) \mid F^\gamma(x, t) = x\}$, i.e. the coincidence set of F^γ and the projection $p: M \times I \to M$. As before, we form $\Phi^\gamma: M \times S^1 \to M$. We may perturb Φ^γ to a smooth (or PL) map Ψ^γ whose image misses ∂M and whose graph meets the graph of the projection p transversely. Then $\mathrm{Fix}(\Psi^\gamma) \equiv \{(x, t) \mid \Psi^\gamma(x, t) = x\}$ is a closed 1-manifold which naturally carries the "intersection orientation", using the order (graph of p, graph of Ψ^γ), as explained, for example, in [DG, § 8 and § 11] and [GN_1, § 6(A)]. This oriented 1-manifold defines an integral 1-cycle, $U(\gamma)$, in $X \times S^1$. The integral homology class determined by this cycle will be called the *intersection class.* If R is a commutative coefficient ring, let $\theta_R(\gamma) \in H_1(M; R)$ be the image of the homology class represented by the cycle $U(\gamma) \otimes 1$ under $p_*: H_1(M \times S^1; R) \to H_1(M; R)$. When $R = \mathbf{Z}$ we write $\theta_{\mathbf{Z}}(\gamma) = \theta(\gamma)$.

Definition C_1. Let R be a commutative ring of coefficients.

$$\chi_1(M; R)(\gamma) = -\theta_R(\gamma) .$$

Definitions A_1, B_1 and C_1 define homomorphisms $\Gamma \to H_1(X; R)$ which are related as follows:

THEOREM 1.1 (Equivalence).

(i) *When R is a principal ideal domain and $H_*(X; R)$ is a free R-module, Definitions A_1 and B_1 agree;*

(ii) *when X is an oriented manifold and R is any commutative coefficient ring, Definitions A_1 and C_1 agree.*

The proof of Theorem 1.1 is deferred until § 10 so as not to interrupt the development of the χ_1-invariant. It is a technical proof, more or less independent of everything else in the paper.

Suppose that $h: X \to Y$ is homotopy equivalence where Y is a finite CW complex. Let $h^{-1}: Y \to X$ be a homotopy inverse for h. Then the map $h_{\#}: X^X \to Y^Y$ given by $f \mapsto hfh^{-1}$ is a homotopy equivalence. In particular, $h_\#$ induces an isomorphism $(h_\#)_*: \Gamma \cong \Gamma' \equiv \pi_1(Y^Y, \mathrm{id})$. The assertion that $\chi_1(X; R)$ is a "homotopy invariant" means that the diagram:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\chi_1(X;R)} & H_1(X;R) \\ {\scriptstyle (h_\#)_*}\downarrow & & \downarrow{\scriptstyle h_*} \\ \Gamma' & \xrightarrow{\chi_1(Y;R)} & H_1(Y;R) \end{array}$$

is commutative. Note that the vertical arrows are isomorphisms.

THEOREM 1.2. $\chi_1(X; R)$ *is a homotopy invariant.*

For the proof, see Corollary 2.10. Theorem 1.2 allows us to extend the definition of $\chi_1(X; R)$ to any topological space X which is homotopy equivalent to a finite complex.

Let $\mathscr{E}(X) \subset X^X$ be the subset of self homotopy equivalences of X and $\mathscr{E}(X, v) \subset \mathscr{E}(X)$ consist of those homotopy equivalences which fix v. There is an *evaluation fibration* $\mathscr{E}(X, v) \hookrightarrow \mathscr{E}(X) \xrightarrow{\eta} X$, where $\eta(f) = f(v)$. The homotopy exact sequence of this fibration yields the exact sequence:

$$\pi_1(\mathscr{E}(X, v), \mathrm{id}) \to \Gamma \xrightarrow{\eta_\#} G \to \pi_0(\mathscr{E}(X, v)) \to \pi_0(\mathscr{E}(X))$$

where $\Gamma \equiv \pi_1(X^X, \mathrm{id}) = \pi_1(\mathscr{E}(X), \mathrm{id})$ and $G = \pi_1(X, v)$. The group $\mathscr{G}(X) \equiv \eta_\#(\Gamma)$ is called the *Gottlieb subgroup* of G.

Gottlieb showed ([Got, Theorem I.4]) that $\mathscr{G}(X)$ lies in the subgroup consisting of those elements of G which act trivially on $\pi_n(X, v)$, for all $n \geqslant 1$; in particular, $\mathscr{G}(X) \subset Z(G)$, the center of G. Indeed by elementary obstruction theory one obtains (see [Got]):

PROPOSITION 1.3. *If* X *is aspherical then* $\mathscr{G}(X) = Z(G)$ *and* $\eta_\#: \Gamma \to Z(G)$ *is an isomorphism.*

In view of this, we will often identify Γ with $Z(G)$ when X is aspherical. (The example of $X = S^2$ shows that the kernel of $\eta_\#: \Gamma \to \mathscr{G}(X)$ may be nontrivial when X not aspherical.)

A group G *is of type* $\mathscr{F}$ if there exists a $K(G, 1)$ which is a finite complex. By Theorem 1.2, the first order Euler characteristic is a homotopy invariant. In particular, applying these definitions to any finite $K(G, 1)$ complex we obtain the *first order Euler characteristic of the group* G *of type* $\mathscr{F}$. For any commutative ring G of coefficients, it is a homomorphism $\chi_1(G; R): Z(G) \to G_{\mathrm{ab}} \otimes R$.

PROPOSITION 1.4. *Let G be of type $\mathscr{F}$. If $\chi(G) \neq 0$ then $\chi_1(G; R)$ is trivial for any coefficient ring R.*

Proof. The center, $Z(G)$, is trivial, by [Got, Theorem IV.1]. Indeed, a short proof of this fact is included below as Proposition 2.4. □

We end this section with the promised fourth definition of $\chi_1(X, R)$ in terms of the transfer maps of [BG], [D_3]. For $\gamma \in \Gamma$, consider $\Phi^\gamma : X \times S^1 \to X$ as above. This defines $\bar{\Phi}^\gamma : X \times S^1 \to X \times S^1$ by $\bar{\Phi}^\gamma(x, z) = (\Phi^\gamma(x, z), z)$ which is a fiber map with respect to the trivial fibration $X \to X \times S^1 \to S^1$. There is an associated S-map (the *transfer*) $\tau(\bar{\Phi}^\gamma) : \Sigma^\infty S^1_+ \to \Sigma^\infty (X \times S^1)_+$. Here, the subscript "+" indicates union with a disjoint basepoint and "Σ^∞" denotes the suspension spectrum of a space. The S-map $\tau(\bar{F})$ induces a homomorphism in homology $\tau(\bar{\Phi}^\gamma)_* : H_*(S^1; R) \to H_*(X \times S^1; R)$.

THEOREM 1.5. *Let R be a field. Then $\chi_1(X; R) = -p_* \tau(\bar{\Phi}^\gamma)_*([S^1])$.* □

This is proved in § 10.

2. DISCUSSION OF DEFINITION A_1

To explain where Definition A_1 comes from, we must review some basic facts about Hochschild homology. Then we show that the formula in Definition A_1 is well-defined and homotopy invariant.

Let R be a commutative ground ring and let S be an associative R-algebra with unit. If M is an $S-S$ bimodule (i.e. a left and right S-module satisfying $(s_1 m)s_2 = s_1(ms_2)$ for all $m \in M$, and $s_1, s_2 \in S$), the *Hochschild chain complex* $\{C_*(S, M), d\}$ consists of $C_n(S, M) = S^{\otimes n} \otimes M$ where $S^{\otimes n}$ is the tensor product of n copies of S and

$$\begin{aligned} d(s_1 \otimes \cdots \otimes s_n \otimes m) &= s_2 \otimes \cdots \otimes s_n \otimes ms_1 \\ &\quad + \sum_{i=1}^{n-1} (-1)^i s_1 \otimes \cdots \otimes s_i s_{i+1} \otimes \cdots \otimes s_n \otimes m \\ &\quad + (-1)^n s_1 \otimes \cdots \otimes s_{n-1} \otimes s_n m \,. \end{aligned}$$

The tensor products are taken over R. The n-th homology of this complex is the n-th *Hochschild homology of S with coefficient bimodule M.* It is denoted by $HH_n(S, M)$. If $M = S$ with the standard $S-S$ bimodule structure then we write $HH_n(S)$ for $HH_n(S, M)$.

We will be concerned mainly with HH_1 and HH_0 which are computed from

$$\begin{aligned} \cdots \to S \otimes S \otimes M &\xrightarrow{d} S \otimes M \xrightarrow{d} M \\ s_1 \otimes s_2 \otimes m &\mapsto s_2 \otimes ms_1 - s_1 s_2 \otimes m + s_1 \otimes s_2 m \\ s \otimes m &\mapsto ms - sm \end{aligned}$$

Next, we consider traces in Hochschild homology. If A is a square matrix over M, we interpret its trace $\sum_i A_{ii}$ as an element of M (i.e. as a Hochschild 0-cycle). The corresponding homology class is denoted by $T_0(A) \in HH_0(S, M)$. If $A^i, i = 1, \ldots, n$, are $q_i \times q_{i+1}$ matrices over S and B is a $q_{n+1} \times q_1$ matrix over M, we define $A^1 \otimes \cdots \otimes A^n \otimes B$ to be the $q_1 \times q_1$ matrix with entries in the R-module $S^{\otimes n} \otimes M$ given by

$$(A^1 \otimes \cdots \otimes A^n \otimes B)_{ij} = \sum_{k_2, \ldots, k_{n+1}} A^1_{i,k_2} \otimes A^2_{k_2,k_3} \otimes \cdots \otimes A^n_{k_n,k_{n+1}} \otimes B_{k_{n+1},j} .$$

The *trace* of $A^1 \otimes \cdots \otimes A^n \otimes B$, written $\mathrm{trace}(A^1 \otimes \cdots \otimes A^n \otimes B)$, is

$$\sum_{k_1, k_2, \ldots, k_{n+1}} A^1_{k_1,k_2} \otimes A^2_{k_2,k_3} \otimes \cdots \otimes A^n_{k_n,k_{n+1}} \otimes B_{k_{n+1},k_1} .$$

which we interpret as a Hochschild n-chain. Observe that the 1-chain $\mathrm{trace}(A \otimes B)$ is a cycle if and only if $\mathrm{trace}(AB) = \mathrm{trace}(BA)$, in which case we denote its homology class by $T_1(A \otimes B) \in HH_1(S, M)$. In the application below, S will be a groupring over the ground ring R and $M = S$.

We will use the notation G_1 for the set of conjugacy classes of a group G. The partition of G into the union of its conjugacy classes induces a direct sum decomposition of $HH_*(\mathbf{Z}G)$ as follows: each generating chain $c = g_1 \otimes \cdots \otimes g_n \otimes m$ can be written in *canonical form* as $g_1 \otimes \cdots \otimes g_n \otimes g_n^{-1} \cdots g_1^{-1} g$ where we think of $g = g_1 \cdots g_n m \in G$ as "marking" the conjugacy class $C(g)$. All the generating chains occurring in the boundary $d(c)$ are easily seen to have markers in $C(g)$ when put into canonical form. For $C \in G_1$ let $C_*(\mathbf{Z}G)_C$ be the subgroup of $C_*(\mathbf{Z}G)$ generated by those generating chains whose markers lie in C. The decomposition $\mathbf{Z}G \cong \oplus_{C \in G_1} \mathbf{Z}C$ as a direct sum of abelian groups determines a decomposition of chain complexes $C_*(\mathbf{Z}G) \cong \oplus_{C \in G_1} C_*(\mathbf{Z}G)_C$. There results a natural isomorphism $HH_*(\mathbf{Z}G) \cong \oplus_{C \in G_1} HH_*(\mathbf{Z}G)_C$ where the summand $HH_*(\mathbf{Z}G)_C$ corresponds to the homology classes of Hochschild cycles marked by the elements of C. We call this summand the *C-component.* Given any $\mathbf{Z}G$-$\mathbf{Z}G$ bimodule N let $\bar{N}$ be the left $\mathbf{Z}G$ module whose underlying abelian group is N and whose left module structure is given by $gm = g \cdot m \cdot g^{-1}$. There is a natural isomorphism $HH_*(\mathbf{Z}G, N) \cong H_*(G, \bar{N})$

which is induced from an isomorphism of the Hochschild complex to the bar complex for computing group homology; see [I, Theorem 1.d]. The decomposition $\overline{\mathbf{Z}G} \cong \oplus_{C \in G_1} \mathbf{Z}C$ is a direct sum of left $\mathbf{Z}G$ modules, inducing a direct sum decomposition $H_*(G, \overline{\mathbf{Z}G}) \cong \oplus_{C \in G_1} H_*(G, \mathbf{Z}C)$. Choosing representatives $g_C \in C$ we have an isomorphism of left $\mathbf{Z}G$ modules $\mathbf{Z}C \cong \mathbf{Z}(G/Z(g_C))$ where $Z(h) = \{g \in G \mid h = ghg^{-1}\}$ denotes the centralizer of $h \in G$. Since $H_*(G, \mathbf{Z}(G/Z(g_C)))$ is naturally isomorphic to $H_*(Z(g_C))$, we obtain a natural isomorphism $HH_*(\mathbf{Z}G) \cong \oplus_{C \in G_1} H_*(Z(g_C))$; furthermore, $HH_*(\mathbf{Z}G)_C$ corresponds to the summand $H_*(Z(g_C))$ under this identification. In particular $HH_0(\mathbf{Z}G) \cong \mathbf{Z}G_1$, the free abelian group generated by the conjugacy classes, and $HH_1(\mathbf{Z}G) \cong \oplus_{C \in G_1} H_1(Z(g_C))$, the direct sum of the abelianizations of the centralizers. Indeed, if $g \otimes g^{-1}g_C$ is a cycle then its homology class in $HH_1(\mathbf{Z}G)$ corresponds to $\{g\} \in H_1(Z(g_C))$.

The augmentation $\varepsilon: \mathbf{Z}G \to \mathbf{Z}$ can be viewed as a morphism of $\mathbf{Z}G$-$\mathbf{Z}G$ bimodules, where $\mathbf{Z}$ is given the trivial bimodule structure, or as a morphism $\varepsilon: \overline{\mathbf{Z}G} \to \bar{\mathbf{Z}}$ of left $\mathbf{Z}G$-modules. Then there is an induced chain map $C_*(\mathbf{Z}G, \mathbf{Z}G) \xrightarrow{\varepsilon} C_*(\mathbf{Z}G, \mathbf{Z})$ and a commutative diagram:

$$\begin{array}{ccc} HH_*(\mathbf{Z}G, \mathbf{Z}G) & \xrightarrow{\varepsilon} & HH_*(\mathbf{Z}G, \mathbf{Z}) \\ \mu \downarrow & & \mu \downarrow \\ H_*(G, \overline{\mathbf{Z}G}) & \xrightarrow{\varepsilon} & H_*(G, \bar{\mathbf{Z}}) \end{array}$$

where the vertical arrows are isomorphisms.

Recall the abelianization homomorphism $A: \mathbf{Z}G \to G_{\mathrm{ab}} = H_1(X) = H_1(G)$ used in Definition A_1.

Proposition 2.1. *If* $\sum_i c_i \otimes n_i \in C_1(\mathbf{Z}G, \mathbf{Z})$ *is a Hochschild* 1*-cycle representing* $z \in HH_1(\mathbf{Z}G, \mathbf{Z})$, *where* $c_i \in \mathbf{Z}G$ *and* $n_i \in \mathbf{Z}$, *then* $\mu(z) = \sum_i A(c_i n_i) \in H_1(G)$.

Proof. This follows from the fact that $d: \mathbf{Z}G \otimes \mathbf{Z}G \otimes \mathbf{Z} \to \mathbf{Z}G \otimes \mathbf{Z}$ becomes $g_1 \otimes g_2 \otimes 1 \mapsto (g_2 - g_1 g_2 + g_1) \otimes 1$. One easily shows that the map $g \otimes 1 \mapsto A(g)$ induces μ. $\square$

With notation as in §1, let $\tilde{D}_k^\gamma: C_k(\tilde{X}) \to C_{k+1}(\tilde{X})$ be the lift of D_k^γ. Write $\tilde{\partial} = \oplus_k \tilde{\partial}_k$, $\tilde{D}^\gamma = \oplus(-1)^{k+1}\tilde{D}_k^\gamma$ and $\tilde{I} = \oplus_k(-1)^k \mathrm{id}_k$ (viewed as matrices). The chain homotopy relation becomes $\tilde{D}^\gamma\tilde{\partial} - \tilde{\partial}\tilde{D}^\gamma = \tilde{I}(1 - \eta_\#(\gamma)^{-1})$ [Explanation: the minus sign occurs on the left because of the sign convention built into the matrix $\tilde{D}^\gamma$; the right hand side is thus because the 0-end of the homotopy F^γ is lifted to the identity, while the 1-end is lifted to the covering translation corresponding to $\eta_{\#}(\gamma)$; the inversion occurs because we have G acting on the right.]

PROPOSITION 2.2. *$\chi_1(X;R)(\gamma)$, as given in Definition A_1, is independent of the choice of the cellular homotopy F^γ representing γ.*

Proof. It is enough to consider the case $R = \mathbf{Z}$. We must show that if $F_1^\gamma \simeq F_2^\gamma : X \times I \to X \operatorname{rel} X \times \{0, 1\}$, with corresponding chain homotopies $D_*^{1,\gamma}$ and $D_*^{2,\gamma}$, then $A(\operatorname{trace}(\tilde{\partial} D^{1,\gamma})) = A(\operatorname{trace}(\tilde{\partial} D^{2,\gamma}))$.

There is a degree 2 chain homotopy $\tilde{E}_k : C_k(\tilde{X}) \to C_{k+2}(\tilde{X})$ such that $\tilde{E}_{k-1}\tilde{\partial}_k - \tilde{\partial}_{k+2}\tilde{E}_k = \tilde{D}^\gamma_{1,k} - \tilde{D}^\gamma_{2,k}$. Write $\tilde{E} = \oplus_k(-1)^{k+2}\tilde{E}_k$ (viewed as a matrix). Then $\tilde{E}\tilde{\partial} + \tilde{\partial}\tilde{E} = \tilde{D}_1^\gamma - \tilde{D}_2^\gamma$. So $\operatorname{trace}(\tilde{\partial} \otimes (\tilde{D}_1^\gamma - \tilde{D}_2^\gamma)) = d\operatorname{trace}(\tilde{\partial} \otimes \tilde{\partial} \otimes \tilde{E})$ is a Hochschild boundary. The desired result now follows from Proposition 2.1. □

Direct calculation yields:

$$d(\operatorname{trace}(\tilde{\partial} \otimes \tilde{D}^\gamma)) = \chi(X)(1 - \eta_{\#}(\gamma)^{-1}) . \tag{2.3}$$

This leads to a quick proof (translating an idea of Stallings [St]) of an important theorem of Gottlieb [Got, Theorem IV.1]:

PROPOSITION 2.4. *If $\chi(X) \neq 0$ then $\mathcal{G}(X)$ is trivial.*

Proof. Since $\chi(X) \neq 0$, (2.3) shows that every $(1 - \eta_{\#}(\gamma)^{-1})$ represents $0 \in HH_0(\mathbf{Z}G)$. This implies that $\eta_{\#}(\gamma) = 1$. □

PROPOSITION 2.5. *In the Hochschild complex, $C_*(\mathbf{Z}G, \mathbf{Z}G)$, $\operatorname{trace}(\tilde{\partial} \otimes \tilde{D}^\gamma)$ is a cycle.*

Proof. If $\chi(X) = 0$, use (2.3). If $\chi(X) \neq 0$, use (2.3) and Proposition 2.4. □

Define the *lift* of $\chi_1(\cdot\,;\mathbf{Z})$ to be the function $\tilde{\mathrm{X}}_1(X) : \Gamma \to HH_1(\mathbf{Z}G)$ which takes γ to $T_1(\tilde{\partial} \otimes \tilde{D}^\gamma)$, the homology class of the cycle $\operatorname{trace}(\tilde{\partial} \otimes \tilde{D}^\gamma)$. The proof of Proposition 2.2 shows that this is also independent of the choice of F^γ representing γ.

There is a left action of $Z(G)$ on $HH_*(\mathbf{Z}G)$. At the level of chains it is defined by

$$\omega \cdot (g_1 \otimes \cdots \otimes g_n \otimes m) = g_1 \otimes \cdots \otimes g_n \otimes (m\omega^{-1})$$

where $\omega \in Z(G)$. One easily checks that this action is compatible with d

and hence makes $HH_*(\mathbf{Z}G)$ into a left $Z(G)$-module. The summand $HH_*(\mathbf{Z}G)_C$ is taken by the left action of ω isomorphically onto the summand $HH_*(\mathbf{Z}G)_{C\omega^{-1}}$ where $C\omega^{-1}$ is the conjugacy class $\{g\omega^{-1} \mid g \in C\}$.

Since η maps Γ into $Z(G)$, η defines a left action of Γ on $C_*(\mathbf{Z}G, \mathbf{Z}G)$ and on $HH_1(\mathbf{Z}G)$. By considering lifts of homotopies, we clearly get:

PROPOSITION 2.6. *When* $HH_1(\mathbf{Z}G)$ *is regarded as a left* Γ*-module,* $\tilde{X}_1(X)$ *becomes a derivation; i.e.* $\tilde{X}_1(X)(\gamma_1\gamma_2) = \tilde{X}_1(X)(\gamma_1) + \gamma_1 \cdot \tilde{X}_1(X)(\gamma_2)$. □

Derivations modulo inner derivations yield one-dimensional cohomology; in particular, $\tilde{X}_1(X)$ defines a cohomology class $\tilde{\chi}_1(X) \equiv [\tilde{X}_1(X)] \in H^1(\Gamma, HH_1(\mathbf{Z}G))$.

The derivation $\tilde{X}_1(X)$ depends on the choice of lifts $\tilde{e}$ of the cells e of X (see § 1). However, we have:

PROPOSITION 2.7. *Up to inner derivations,* $\tilde{X}_1(X)$ *is independent of the choice of cell orientations and of the choice of lifts. Hence* $\tilde{\chi}_1(X)$ *is a well-defined cohomology class.*

Proof. Another choice of cell orientations and lifts to the universal cover determines a chain complex $(C'_*(\tilde{X}), \tilde{\partial}'_*)$ and a chain homotopy $\tilde{E}^{\gamma}_k : C'_k(\tilde{X}) \to C'_{k+1}(\tilde{X})$. By the "change of basis formula", [GN$_1$, Proposition 3.3], we have:

$$T_1(\tilde{\partial}' \otimes \tilde{E}^{\gamma}) - T_1(\tilde{\partial} \otimes \tilde{D}^{\gamma}) = T_1(U \otimes U^{-1}(1 - \eta_{\#}(\gamma)^{-1}))$$

where U is the change of basis matrix. Since $\gamma \mapsto T_1(U \otimes U^{-1}(1 - \eta_{\#}(\gamma)^{-1}))$ is clearly an inner derivation, the conclusion follows. □

We may regard Definition A$_1$ as defining a cohomology class $\chi_1(X) \in H^1(\Gamma, H_1(G))$. Clearly we have:

PROPOSITION 2.8. *Under the homomorphism induced by* $\varepsilon_* : HH_1(\mathbf{Z}G) \to H_1(G)$, $\tilde{\chi}_1(X)$ *is taken to* $\chi_1(X)$. *Thus Definition* A_1 *is independent of the choice of lifts and* $\chi_1(X)$ *is homomorphism.* □

Despite Propositions 2.2 and 2.8, the formula in Definition A$_1$ might appear to depend on the CW structure of X. However, we have:

THEOREM 2.9. *The cohomology classes* $\tilde{\chi}_1(X)$ *and* $\chi_1(X)$ *are homotopy invariants.*

Proof. Since $\varepsilon_*(\tilde{\chi}_1(X)) = \chi_1(X)$, it is sufficient to show that $\tilde{\chi}_1(X)$ is a homotopy invariant. Let $X \to Y$ be a homotopy equivalence. By making use of mapping cylinders, we may assume without loss of generality that $X \to Y$ is an inclusion of X into Y as a subcomplex. Choose orientations for the cells of Y and oriented lifts of these cells to the universal cover, $\tilde{Y}$, of Y. Let $\tilde{X} = p^{-1}(X)$ where $p\colon \tilde{Y} \to Y$ is the covering projection. Since $X \hookrightarrow Y$ is a homotopy equivalence, $\tilde{X}$ is the universal cover of X. Choose the basepoint to be a vertex of X. Given $\gamma \in \Gamma' = \pi_1(\mathscr{E}(Y), \mathrm{id})$, the homotopy extension property allows one to find a self homotopy of the identity $F^\gamma\colon Y \times I \to Y$ which has the additional property that $F^\gamma(X \times I) \subset X$. Let $\tilde{D}^\gamma_*\colon C_*(\tilde{Y}) \to C_*(\tilde{Y})$ be the chain homotopy determined by F^γ and let $\tilde{D}^\gamma_* |$ be the restriction of $\tilde{D}^\gamma_*$ to $C_*(\tilde{X})$. Let $C_*(\tilde{Y}, \tilde{X})$ be the relative chain complex with boundary operator denoted by $\bar{\partial}$. Then $\tilde{D}^\gamma_*$ induces a chain homotopy on this complex which we will denote by $\bar{D}^\gamma_*$. There is a commutative diagram:

$$\begin{array}{ccccc}
C_*(\tilde{X}) & \to & C_*(\tilde{Y}) & \to & C_*(\tilde{Y}, \tilde{X}) \\
{\scriptstyle \tilde{D}^\gamma_*|}\downarrow & & {\scriptstyle \tilde{D}^\gamma_*}\downarrow & & {\scriptstyle \bar{D}^\gamma_*}\downarrow \\
C_*(\tilde{Y}) & \to & C_*(\tilde{Y}) & \to & C_*(\tilde{Y}, \tilde{X})\,.
\end{array}$$

By [GN_1, Proposition 3.5], we have that, in $HH_1(\mathbf{Z}G)$:

$$T_1(\tilde{\partial} \otimes \tilde{D}^\gamma) - T_1(\tilde{\partial} | \otimes \tilde{D}^\gamma |) = T_1(\bar{\partial} \otimes \bar{D}^\gamma)\,.$$

Although for a given $\gamma \in \Gamma'$, $T_1(\bar{\partial}_* \otimes \bar{D}^\gamma_*)$ could, in principle, be nonzero we will show that $\gamma \mapsto T_1(\bar{\partial}_* \otimes \bar{D}^\gamma_*)$ is a coboundary. Let $\bar{C}_* = C_*(\tilde{Y}, \tilde{X})$. Since $X \hookrightarrow Y$ is a homotopy equivalence, $\bar{C}$ is a contractible chain complex. Let $H_*\colon \bar{C}_* \to \bar{C}_*$ be a chain contraction. Then $\bar{D}^\gamma_*$ is chain homotopic to $H_*(1 - \eta_\#(\gamma)^{-1})$ via the chain homotopy $H_*(\bar{D}^\gamma_* - H_*(1 - \eta_\#(\gamma)^{-1}))$. Using the given bases, we can represent $\bar{\partial}$ and H as matrices over $\mathbf{Z}\pi_1(Y)$. Reusing symbols, we write $\bar{\partial} = \oplus_i \bar{\partial}_i$, $H = \oplus_i (-1)^{i+1} H_i$ (viewed as matrices). Then, by [GN_1, Lemma 3.2], $T_1(\bar{\partial} \otimes \bar{D}^\gamma) = T_1(\bar{\partial} \otimes H(1 - \eta_\#(\gamma)^{-1}))$ where $H(1 - \eta_\#(\gamma)^{-1})$ is the matrix obtained by multiplying each element of H on the right by $1 - \eta_\#(\gamma)^{-1} \in \mathbf{Z}\pi_1(Y)$. Clearly, $\gamma \mapsto T_1(\bar{\partial} \otimes H(1 - \eta_\#(\gamma)^{-1}))$ is an inner derivation. It follows that the derivations $\gamma \mapsto T_1(\tilde{\partial} \otimes \tilde{D}^\gamma)$ and $\gamma \mapsto T_1(\tilde{\partial} | \otimes \tilde{D}^\gamma |)$ represent the same cohomology class. □

COROLLARY 2.10. *The formula in Definition* A_1 *is a well-defined homotopy invariant of* X. □

3. Some calculations

In this section we give some computations of $\chi_1(X)$ and $\tilde{\chi}_1(X)$ which make use of explicit cell decompositions of the universal cover, $\tilde{X}$, of X. The simplest non-trivial example is the circle, $X = S^1$, which is treated in (A). In (B) we consider aspherical 2-complexes, X, arising from groups with two generators and one defining relation. In (C), X is a 3-dimensional lens space with odd order fundamental group; in fact, the computation there is already implicit in [GN$_1$, §5(B)]. In (D), X is the real projective plane.

(A) Finite graphs

A finite connected 1-complex, X, is aspherical so by Propositions 1.3 and 2.4, $\Gamma = \pi_1(\mathscr{E}(X), \mathrm{id})$ is trivial unless X has the homotopy type of S^1. Take X to be S^1 with one 0-cell, v, and one 1-cell, e. Then $\tilde{X}$ is the real line with the usual CW structure. Orient v by $+1$ and e by $u \mapsto e^{2\pi i u}$. Let $t \in T \equiv \pi_1(S^1, v)$ be represented by the loop $u \mapsto e^{-2\pi i u}$ (this generator of T has been chosen for compatibility with §6). Recall that we use the right action of T, so

$$\tilde{\partial} = \begin{bmatrix} 0 & t-1 \\ 0 & 0 \end{bmatrix}.$$

The matrix $\tilde{D}^{[R_1]}$ corresponding to positive rotation, $R_1 : S^1 \times I \to S^1$, through 2π (the first "tumble" in the language of §6) is

$$\tilde{D}^{[R_1]} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix};$$

note that the Sign Convention of §1 is used here. Thus $\tilde{\mathrm{X}}_1(S^1)([R_1])$ is represented by $(t-1) \otimes 1$ which is homologous to $t \otimes 1$, and $\chi_1(S^1)([R_1]) = \{t\}$. Now, $[R_1]$ generates the infinite cyclic group Γ. Making the standard identifications of Γ and T with $\mathbf{Z}$ (i.e. identifying $[R_1]$ and t^{-1} with $1 \in \mathbf{Z}$), we obtain:

Example 3.1. $\chi_1(S^1) : \mathbf{Z} \to \mathbf{Z}$ is multiplication by -1.

Remark. The circle belongs to the classes of spaces considered in §4 and §6, so the methods there also apply.

(B) Groups with two generators and one relation

Let X be a finite 2-complex with one vertex, v, and one 2-cell, e^2. We further assume that X is aspherical. By Lyndon's theorem [Ly], this is the case if and only if the element of the free group defined by the attaching map of the 2-cell is not a proper power. As in (A), the group $\Gamma \cong Z(\pi_1(X, v))$ is trivial unless X has two 1-cells, e_1^1 and e_2^1 (otherwise $\chi(X) \neq 0$), so we assume this.

The case when X is homotopy equivalent to the 2-torus is exceptional. The following calculation is a special case of Example 6.15. Alternatively, the same result can be obtained by the method of Example 3.8 below. See also Corollary 4.8.

Example 3.2. Let X be homotopy equivalent to the 2-torus. Then $\tilde{\chi}_1(X) = 0$. Consequently, Proposition 2.8 implies $\chi_1(X) = 0$.

In all (aspherical) cases other than the 2-torus, Γ is known to be either trivial or infinite cyclic [Mu].

Orient v by $+1$, and choose orientations for the the other cells. There is a corresponding presentation $\langle x_1, x_2 \mid r \rangle$ of $G = \pi_1(X, v)$, where x_i denotes the element of G represented by the oriented loop e_i^1, and r is the attaching word in $\{x_i^{\pm}\}$ with respect to the chosen orientation on e^2. Choose lifts of the cells so that:

$$\tilde{\partial}_1(\tilde{e}_i^1) = (x_i - 1)\tilde{v} \quad \text{and} \quad \tilde{\partial}_2(\tilde{e}^2) = \frac{\partial r}{\partial x_1}\tilde{e}_1^1 + \frac{\partial r}{\partial x_2}\tilde{e}_2^1 .$$

We have written these in terms of the left action of G because we are using the free differential calculus [B, p. 45] which is traditionally done in terms of left actions. We will then convert to right actions using the involution $*: \mathbf{Z}G \to \mathbf{Z}G$, $\sum_i n_i g_i \mapsto \sum_i n_i g_i^{-1}$.

For $\gamma \in Z(G)$, there is a unique (up to homotopy) cellular homotopy $F^\gamma: \mathrm{id}_X \to \mathrm{id}_X$. The track of the basepoint presents γ as a word in $\{x_i^{\pm}\}$, and

$$\tilde{D}_0^\gamma(\tilde{v}) = -\frac{\partial \gamma}{\partial x_1}\tilde{e}_1^1 - \frac{\partial \gamma}{\partial x_2}\tilde{e}_2^1 .$$

There are $\sigma_1, \sigma_2 \in \mathbf{Z}G$ such that $\tilde{D}_1^\gamma(\tilde{e}_i) = \sigma_i \tilde{e}^2$. Thus the relevant matrices are:

$$\tilde{\partial}_1 = [x_1^{-1} - 1 \;\; x_2^{-1} - 1], \quad \tilde{\partial}_2 = \begin{bmatrix} \left(\frac{\partial r}{\partial x_1}\right)^* \\ \left(\frac{\partial r}{\partial x_2}\right)^* \end{bmatrix}, \quad \tilde{D}_0 = \begin{bmatrix} -\left(\frac{\partial \gamma}{\partial x_1}\right)^* \\ -\left(\frac{\partial \gamma}{\partial x_2}\right)^* \end{bmatrix} .$$

and $\tilde{D}_1 = [\sigma_1^* \;\; \sigma_2^*]$. So $\tilde{\mathrm{X}}_1(X)(\gamma)$ is represented by the chain:

$$(3.3) \quad \mathrm{trace}(\tilde{\partial} \otimes \tilde{D}^\gamma) = \sum_{i=1}^{2} \left[(x_i^{-1} - 1) \otimes \left(\frac{\partial \gamma}{\partial x_i}\right)^* + \left(\frac{\partial r}{\partial x_i}\right)^* \otimes \sigma_i^* \right] .$$

By Proposition 2.1, this implies:

$$\chi_1(X)(\gamma) = \sum_{i=1}^{2} \left[-\varepsilon\left(\frac{\partial\gamma}{\partial x_i}\right) A(x_i) - \varepsilon(\sigma_i) A\left(\frac{\partial r}{\partial x_i}\right)\right]$$

where $\varepsilon: \mathbf{Z}G \to Z$ is augmentation. For any $g \in G$ represented by the word w in $\{x_i^{\pm}\}$, $A(g) = \sum_{j=1}^{2} \varepsilon\left(\frac{\partial w}{\partial x_j}\right) A(x_j)$. Substituting, we get:

$$\chi_1(X)(\gamma) = -A(\gamma) - \sum_{1 \leqslant i,j \leqslant 2} \varepsilon(\sigma_i)\varepsilon\left(\frac{\partial^2 r}{\partial x_j \partial x_i}\right) A(x_j) .$$

The fact that $\tilde{D}\tilde{\partial} - \tilde{\partial}\tilde{D} = \tilde{I}(1 - \eta_{\#}(\gamma)^{-1})$ yields six equations in $\mathbf{Z}G$. It is straightforward to check that when ε is applied to these they reduce to:

LEMMA 3.4. *For all* $1 \leqslant i, j \leqslant 2$, $\varepsilon(\sigma_i)\varepsilon\left(\frac{\partial r}{\partial x_j}\right) = 0.$ □

The chain complex $C_*(X)$ is $\mathbf{Z} \xrightarrow{\partial_2} \mathbf{Z} \oplus \mathbf{Z} \xrightarrow{\partial_1} \mathbf{Z}$ where

$$\partial_2(1) = \left[\varepsilon\left(\frac{\partial r}{\partial x_1}\right), \quad \varepsilon\left(\frac{\partial r}{\partial x_2}\right)\right]$$

and $\partial_1 = 0$. If $H_2(X) = 0$ then $\partial_2 \neq 0$, and by Lemma 3.4, $\varepsilon(\sigma_1) = \varepsilon(\sigma_2) = 0$. Hence:

PROPOSITION 3.5. *If* $H_2(X) = 0$ *then* $\chi_1(X) = -A$. □

If $H_2(X) \neq 0$ then $\partial_2 = 0$. In this case we may regard $A(x_1)$ and $A(x_2)$ as a basis for the free abelian group G_{ab}. Writing $H(r)$ for the *Fox Hessian* matrix of r, namely $H(r)_{ij} = \varepsilon\left(\frac{\partial^2 r}{\partial x_i \partial x_j}\right)$, and $H(r)^t$ for its transpose we have:

PROPOSITION 3.6. *If* $H_2(X) \neq 0$ *then*

$$\chi_1(X)(\gamma) = -A(\gamma) - [\varepsilon(\sigma_1)\ \varepsilon(\sigma_2)] H(r)^t \begin{bmatrix} A(x_1) \\ A(x_2) \end{bmatrix} . \quad \square$$

The matrix $H(r)$ can be computed once we are given the relation r. The integers $\varepsilon(\sigma_1)$ and $\varepsilon(\sigma_2)$ depend on γ; in general, they are hard to compute although we will do so in some special cases (see Examples 3.8 and 3.9 below).

The matrix $H(r)$ is determined by the cup product $H^1(X) \otimes H^1(X) \to H^2(X)$:

PROPOSITION 3.7. *Assume* $H_2(X) \neq 0$. *Let* $\{\bar{A}(x_1), \bar{A}(x_2)\}$ *be the dual basis for* $H^1(X)$. *Then* $H(r)_{ij} = (\bar{A}(x_i) \cup \bar{A}(x_j))([e^2])$; *hence:*

$$\chi_1(X)(\gamma) = -A(\gamma) - (\bar{A}(x_1) \cup \bar{A}(x_2))([e^2])(\varepsilon(\sigma_1)A(x_2) - \varepsilon(\sigma_2)A(x_1)) .$$

Proof. This is the same formula given by Definition B_1 (note that $H_*(X)$ is free abelian and so Definition B_1 applies to integral coefficients). A direct proof of Proposition 3.7 is also possible. □

Example 3.8. $G = \langle x_1, x_2 \,|\, x_2 x_1^m x_2^{-1} x_1^{-m} \rangle$, $m \geqslant 2$. Here, $Z(G)$ is generated by x_1^m, and $H_2(X) \neq 0$. One calculates: $\dfrac{\partial r}{\partial x_1} = (x_2 - 1) \sum_{i=0}^{m-1} x_1^i$, $\dfrac{\partial r}{\partial x_2} = 1 - x_1^m$, $\dfrac{\partial \gamma}{\partial x_1} = \sum_{i=0}^{m-1} x_1^i$, $\dfrac{\partial \gamma}{\partial x_2} = 0$, $\sigma_1 = 0$ and $\sigma_2 = 1$. (Actually, one sees these values for the sigmas intuitively and then one checks that the resulting $\tilde{D}$ gives the right answer.) Thus $\tilde{X}_1(X)(x_1^m)$ is represented by the cycle $(x_1^{-1} - 1) \otimes \sum_{i=0}^{m-1} x_1^{-i} + (1 - x_1^{-m}) \otimes 1$ which is homologous to the canonical form: $x_1^{-1} \otimes x_1 \ (\sum_{i=1}^{m-1} x_1^{-i}) + x_1^{m-1} \otimes x_1^{-(m-1)} x_1^{-m}$. It follows that (see §2) $\tilde{X}_1(X)(x_1^m) \in HH_1(\mathbf{Z}G) \cong \bigoplus_{C \in G_1} H_1(Z(g_C))$ has $[x_1^{-i}]$-summand $-\{x_1\} \in H_1(Z(x_1^{-i}))$, for $1 \leqslant i \leqslant m-1$, and $[x_1^{-m}]$-summand $(m-1)\{x_1\} \in H_1(G) = G_{\mathrm{ab}}$; here, $[g]$ denotes the conjugacy class of g. By Proposition 2.1 (or 3.6), $\chi_1(X)(x_1^m) = 0$. It is not difficult to see that $\tilde{X}_1(X)$ is not an inner derivation. In particular, the first order Euler characteristic is zero, while $\tilde{\chi}_1(X) \neq 0$.

EXAMPLE 3.9. $G = \langle x_1, x_2 \,|\, x_1^m x_2^n \rangle$, $m \neq 0$ and $n \neq 0$. (If m and n are relatively prime, then G is the group of the $(m, -n)$ torus knot.) Here, $Z(G)$ is generated by $x_1^m = x_2^{-n}$, and $H_2(X) = 0$. By Proposition 3.5, $\chi_1(X)(x_1^m) = -mA(x_1) = nA(x_2)$. It is also of interest to calculate $\tilde{X}_1(X)(x_1^m)$. We get $\dfrac{\partial r}{\partial x_1} = \sum_{i=0}^{m-1} x_1^i$, $\dfrac{\partial r}{\partial x_2} = x_1^m \sum_{i=0}^{n-1} x_2^i$, $\dfrac{\partial \gamma}{\partial x_1} = \sum_{i=0}^{m-1} x_1^i$, $\dfrac{\partial \gamma}{\partial x_2} = 0$, $\sigma_1 = 0$ and $\sigma_2 = x_2 - 1$. Thus $\tilde{X}_1(X)(x_1^m)$ is represented by the cycle $(x_1^{-1} - 1) \otimes \sum_{i=0}^{m-1} x_1^{-i} + (\sum_{i=0}^{n-1} x_2^{-i}) x_1^{-m} \otimes (x_2^{-1} - 1)$ which is homologous to the canonical form:

$$\sum_{i=1}^{m-1} (x_1^{-1} \otimes x_1 x_1^{-i}) + \sum_{i=1}^{n-1} (x_2 \otimes x_2^{-1} x_2^i) + x_1^{m-1} \otimes x_1^{-(m-1)} x_1^{-m}$$
$$+ x_2 \otimes x_2^{-1} 1 .$$

(C) LENS SPACES

Let (p, q) be a pair of relatively prime positive integers with $p > 1$. The lens space $L(p, q)$ is the orbit space of the action of the cyclic group $\mathbf{Z}/p = \langle x \mid x^p = 1 \rangle$ on the 3-sphere $S^3 = \{(z_0, z_1) \in \mathbf{C}^2 \mid |z_0|^2 + |z_1|^2 = 1\}$ defined by $x(z_0, z_1) = (e^{2\pi i/p} z_0, e^{2\pi i q/p} z_1)$. The point in $L(p, q)$ determined by the orbit of $(z_0, z_1) \in S^3$ will be denoted $[z_0, z_1]$.

For any pair of integers (m, n) such that $m = n \bmod p$ define a smooth S^1 action $\gamma_{m,n}: S^1 \times L(p, q) \to L(p, q)$ by $e^{2\pi i\theta}[z_0, z_1] = [e^{2\pi i\theta m/p} z_0, e^{2\pi i\theta nq/p} z_1]$. These actions represent elements of $\Gamma = \pi_1(\mathscr{E}(L(p, q)), \mathrm{id})$.

The group $HH_1(\mathbf{Z}[\mathbf{Z}/p])$ is isomorphic to a direct sum of p copies of $\mathbf{Z}/p$; furthermore, the Hochschild 1-cycles $\{x \otimes x^{-1-k} \mid k = 0, \dots, p-1\}$ project to a set of generators for $HH_1(\mathbf{Z}[\mathbf{Z}/p])$. Define $c_i, d_i \in \mathbf{Z}$ for $0 \leqslant i \leqslant p-1$ by $m - i - 1 = (c_i - 1)p + b_i$ and $nq - i - 1 = (d_i - 1)p + b_i'$ where $0 \leqslant b_i, b_i' \leqslant p - 1$. Let $s_k = c_{k-1} + r d_{kq-1}$, where the indices are interpreted mod p and $rq = 1 \bmod p$.

There is a natural cell structure on the universal cover, S^3, of $L(p, q)$ (see [GN_1, §5(B)]). Using this cell structure, [GN_1, Lemma 5.3] asserts:

PROPOSITION 3.10. *$\tilde{\chi}_1(L(p, q))([\gamma_{m,n}]) \in HH_1(\mathbf{Z}[\mathbf{Z}/p])$ is represented by the Hochschild cycle* $-\sum_{k=0}^{p-1} s_k x \otimes x^{-1-k}$. □

Remark. We take this opportunity to correct some inadvertently omitted minus signs from the computed examples in [GN_1, §5]. In order to conform with our Sign Convention (see §1) used both here and in [GN_1], the various chain homotopies $\tilde{D}$ appearing in the explicit computations of [GN_1, §5] should be replaced by $-\tilde{D}$. Consequently, in [GN_1, Lemma 5.3], [GN_1, Proposition 5.4] and [GN_1, Corollary 5.5] $\beta(\gamma_{m,n})$, $R(\gamma_{m,n})$ and $L(\gamma_{m,n})$ should be replaced by $-\beta(\gamma_{m,n})$, $-R(\gamma_{m,n})$ and $-L(\gamma_{m,n})$ respectively. Similarly, $R(F_n)$ should be replaced by $-R(F_n)$ in [GN_1, Theorem 5.1] and $R(\Phi_2)$ should be replaced by $-R(\Phi_2)$ in [GN_1, §5(C)].

The homomorphism $\varepsilon: HH_1(\mathbf{Z}[\mathbf{Z}/p]) \to H_1(\mathbf{Z}/p)$ takes the generators $\{x \otimes x^{-1-k}\}$ to the same generator, α, of $H_1(\mathbf{Z}/p)$. From the proof of [GN_1, Corollary 5.5], we deduce:

PROPOSITION 3.11. $\chi_1(L(p, q))([\gamma_{m,n}]) = -(m + n)\alpha$. □

If p is odd then Propositions 3.10 and 3.11 give complete computations of $\tilde{\chi}_1(L(p, q))$ and $\chi_1(L(p, q))$ respectively because the $[\gamma_{m,n}]$'s generate Γ;

indeed by [GN_1, Proposition 5.7], for odd p, Γ is cyclic of order $2p^2$. The proof there also shows that $2[\gamma_{1,1}]$ is of order p^2 and that $p[\gamma_{0,p}]$ is of order 2 in Γ, so $[\gamma_{2,2+p^2}]$ generates Γ.

(D) THE PROJECTIVE PLANE

We saw that when X is aspherical and $\chi(X) \neq 0$ then $\Gamma = 0$ and so our first order invariants vanish. In the presence of non-trivial higher homotopy these invariants need not vanish, despite $\chi(X) \neq 0$, as demonstrated by the example of the real projective plane $X = P^2$.

Write $G \equiv \pi_1(P^2) \cong \mathbf{Z}/2$; denote the generator of G by t. Give P^2 the customary cell structure consisting of one cell in each of dimensions 0, 1, and 2. The universal cover $\tilde{P}^2$ is naturally identified with S^2 and the corresponding cellular chain complex is:

$$C_2(S^2) \xrightarrow{1+t^{-1}} C_1(S^2) \xrightarrow{t^{-1}-1} C_0(S^2) .$$

Every element of Γ can be represented by a basepoint preserving homotopy $F: P^2 \times I \to P^2$ with $F_0 = F_1 = \mathrm{id}_{P^2}$. We have $\tilde{F}_0 = \tilde{F}_1 = \mathrm{id}_{S^2}$ because the basepoint is preserved. It is easy to verify that the corresponding chain homotopy $\tilde{D}_*: C_*(S^2) \to C_*(S^2)$ is then zero on $C_0(S^2)$ and takes $\tilde{e}_1$ to $\tilde{e}_2 m(1-t^{-1})$ where $m \in \mathbf{Z}$. By elementary obstruction theory, there exists $F \equiv F^{(m)}$ realizing any $m \in \mathbf{Z}$. In this case $\mathrm{trace}(\tilde{\partial} \otimes \tilde{D}) = (1+t^{-1}) \otimes m(1-t^{-1})$ which is homologous to the canonical form $mt^{-1} \otimes tt^{-1} - mt^{-1} \otimes tt^{-2}$. Since $\chi(P^2) = 1 \neq 0$, the Gottlieb group $\eta_\#(\Gamma) \equiv \mathscr{G}(P^2) = 0$ and so the derivation $\tilde{X}_1(P^2)$ is a homomorphism and need not be distinguished from its cohomology class $\tilde{\chi}_1(P^2) \in H^1(\Gamma, HH_1(\mathbf{Z}(\mathbf{Z}/2))) \cong \mathrm{Hom}(\Gamma, HH_1(\mathbf{Z}(\mathbf{Z}/2)))$. It follows that

$$\tilde{\chi}_1(P^2)\,([F^{(m)}]) = (m, -m) \in \mathbf{Z}/2 \oplus \mathbf{Z}/2 \cong HH_1(\mathbf{Z}(\mathbf{Z}/2)) .$$

In particular, when m is odd $\tilde{\chi}_1(P^2)\,([F^{(m)}]) \neq 0$. On the other hand, this shows $\chi_1(P^2) = 0$.

4. S^1-FIBRATIONS

In this section we investigate the first order Euler characteristic of the total space of an orientable Serre fibration with S^1-fiber.

Let $S^1 \to X \xrightarrow{\pi} B$ be an orientable Serre fibration where B is a (not necessarily finite) connected CW complex and X has the homotopy type of a finite complex. By classical obstruction theory, fiber homotopy equivalence classes of orientable S^1-fibrations over a CW complex B are classified by the integral cohomology group $H^2(B;\mathbf{Z})$. Given an element $e \in H^2(B;\mathbf{Z}) \cong [B, \mathbf{C}P^\infty]$ one obtains a principal $U(1)$-bundle over B by pulling back, via a continuous map $B \to \mathbf{C}P^\infty$ representing e, the $U(1)$-bundle associated to the canonical complex line bundle over the infinite dimensional complex projective space $\mathbf{C}P^\infty$. Thus we can assume, without loss of generality, that $S^1 \to X \xrightarrow{\pi} B$ is a principal $U(1)$-bundle. In particular, there is a free $U(1)$-action on X which we will write as $\Phi: X \times S^1 \to X$. Let $\tau \in \Gamma \equiv \pi_1(\mathscr{E}(X), 1)$ be the element represented by Φ ($\Phi = \Phi^\tau$ in the notation of § 1). For any coefficient ring R, let $\{r\} \in H_1(X;R)$ denote the image of τ under the composite:

$$\Gamma \xrightarrow{\eta} \pi_1(X) \to H_1(X) \to H_1(X;R)\ .$$

Also, let e_R be the image of the element $e \in H^2(B;\mathbf{Z})$ which classifies $S^1 \to X \xrightarrow{\pi} B$ under the homomorphism $H^2(B;\mathbf{Z}) \to H^2(B;R)$.

LEMMA 4.1. *If* $\mathbf{F}$ *is a field, then* $\{\tau\} \in H_1(X;\mathbf{F})$ *is non-zero if and only if* $e_\mathbf{F} = 0$.

Proof. Consider the Gysin homology sequence for the fibration $S^1 \to X \xrightarrow{\pi} B$:

$$\cdots \to H_2(B;\mathbf{F}) \xrightarrow{e_\mathbf{F}\cap} H_0(B;\mathbf{F}) \xrightarrow{\theta_0} H_1(X;\mathbf{F}) \xrightarrow{\pi_*} H_1(B;\mathbf{F}) \to 0\ .$$

Since $H_2(B;\mathbf{F}) \xrightarrow{e_\mathbf{F}\cap} H_0(B;\mathbf{F}) \cong \mathbf{F}$ is just evaluation of the cohomology class $e_\mathbf{F}$ on homology, θ_0 is non-zero if and only if $e_\mathbf{F} = 0$. Let $\upsilon \in X$ be a basepoint and let $\{\pi(\upsilon)\} \in H_0(B;\mathbf{F})$ be the generator determined by the inclusion of $\pi(\upsilon)$ into B. The fact that $\theta_0(\{\upsilon\}) = \{\tau\}$ follows from the naturality of the Gysin sequence homology sequence, by mapping the Gysin sequence of the trivial fibration $S^1 \to S^1 \to \pi(\upsilon)$, via the homomorphism induced by inclusion, into the Gysin sequence for $S^1 \to X \xrightarrow{\pi} B$. □

THEOREM 4.2. *Let* $\mathbf{F}$ *be a field. If* $e_\mathbf{F} \neq 0$ *then* $\chi_1(X;\mathbf{F})(\tau) = 0$. *If* $e_\mathbf{F} = 0$ *then* $H_*(B;\mathbf{F})$ *is finite dimensional over* F *and* $\chi_1(X;\mathbf{F})(\tau) = -\chi(B;\mathbf{F})\{\tau\}$ *where* $\chi(B;\mathbf{F}) = \sum_{i\geqslant 0}(-1)^i \dim_\mathbf{F} H_i(B;\mathbf{F})$.

Proof. In this proof, all homology and cohomology groups will have coefficients in the field $\mathbf{F}$. Since B is the orbit space of the $U(1)$-action on X given by Φ, there is a commutative square:

$$\begin{array}{ccc} X \times S^1 & \xrightarrow{\Phi} & X \\ {\scriptstyle \pi\times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \pi} \\ B \times S^1 & \xrightarrow{p} & B \end{array}$$

where $p: B \times S^1 \to B$ is projection. This square induces a commutative ladder mapping the Gysin homology sequence of $S^1 \to X \times S^1 \xrightarrow{\pi \times \mathrm{id}} B \times S^1$ to the Gysin homology sequence of $S^1 \to X \xrightarrow{\pi} B$:

$$\begin{tikzcd}
H_i(B \times S^1) \arrow[r, "\theta'"] \arrow[d, "p_*"] & H_{i+1}(X \times S^1) \arrow[r, "(\pi\times \mathrm{id})_*"] \arrow[d, "\Phi_*"] & H_{i+1}(B \times S^1) \arrow[r] \arrow[d, "p_*"] & H_{i-1}(B \times S^1) \arrow[d, "p_*"] \\
H_i(B) \arrow[r, "\theta"] & H_{i+1}(X) \arrow[r, "\pi_*"] & H_{i+1}(B) \arrow[r, "e_{\mathbf{F}} \cap"] & H_{i-1}(B)
\end{tikzcd}$$

For each integer $0 \leqslant i \leqslant \dim X$ choose a basis $\{b_1^i, \ldots, b_{\beta_i}^i\}$ for $H_i(X)$ such that for some integer $m_i \leqslant \beta_i \{b_{m_i+1}^i, \ldots, b_{\beta_i}^i\}$ is a basis for the kernel of $\pi_*: H_i(X) \to H_i(B)$. The corresponding dual basis for $H^i(X)$ will be denoted by $\{\bar{b}_1^i, \ldots, \bar{b}_{\beta_i}^i\}$. Since we are using coefficients in a field, we make the identifications $H_*(B \times S^1) \cong H_*(B) \otimes H_*(S^1)$ and $H_*(X \times S^1) \cong H_*(X) \otimes H_*(S^1)$ via the natural isomorphism given by the homology exterior product. Let $u \in H_1(S^1)$ be the generator determined by the standard orientation of S^1. Using Definition B_1,

$$\chi_1(X; \mathbf{F})(\tau) = \sum_{k \geqslant 0} (-1)^{k+1} \sum_{j=1}^{\beta_k} \bar{b}_j^k \cap \Phi_*(b_j^k \otimes u)\,.$$

Consider $b_j^i \otimes u \in H_{i+1}(X \times S^1)$ where $m_i + 1 \leqslant j \leqslant \beta_i$. Since b_j^i lies in $\ker \pi_*$, the exactness of the Gysin sequence implies that $b_j^i \otimes u = \theta'(c \otimes u)$ for some $c \in H_i(B)$. Consequently,

$$\Phi_*(b_j^i \otimes u) = \Phi_*(\theta'(c \otimes u)) = \theta(p_*(c \otimes u)) = 0$$

because $p_*(c \otimes u) = 0$. It follows that

$$\chi_1(X; \mathbf{F})(\tau) = \sum_{k \geqslant 0} (-1)^{k+1} \sum_{j=1}^{m_k} \bar{b}_j^k \cap \Phi_*(b_j^k \otimes u)\,. \tag{4.3}$$

For each k, the set $\{\pi_*(b_1^k), \ldots, \pi_*(b_{m_k}^k)\}$ is a basis for the image of $\pi_*: H_k(X) \to H_k(B)$. Extend this set (in any manner) to basis for $H_k(B)$ and let $\{\overline{\pi_*(b_1^k)}, \ldots, \overline{\pi_*(b_{m_k}^k)}\}$ denote the corresponding portion of the dual basis for $H^k(B)$. Then $\bar{b}_j^k = \pi^*\overline{(\pi_*(b_j^k))}, 0 \leqslant j \leqslant m_k$. Consider the commutative diagram:

$$\begin{tikzcd}
H^k(B \times S^1) \arrow[r, "(\pi\times \mathrm{id})^*"] & H^k(X \times S^1) \\
H^k(B) \arrow[u, "p^*"] \arrow[r, "\pi^*"] & H^k(X)\,. \arrow[u, "\Phi^*"]
\end{tikzcd}$$

Then, for $0 \leqslant j \leqslant m_k$,

$$
\begin{aligned}
\bar{b}_j^k \cap \Phi_*(b_j^k \otimes u) &= \Phi_*(\Phi^*(\bar{b}_j^k) \cap (b_j^k \otimes u)) \\
&= \Phi_*(\Phi^*(\pi^*\overline{(\pi_*(b_j^k))}) \cap (b_j^k \otimes u)) \\
&= \Phi_*((\pi \times \mathrm{id})^*(p^*\overline{(\pi_*(b_j^k))}) \cap (b_j^k \otimes u)) \\
&\quad \text{using the above diagram} \\
&= \Phi_*((\bar{b}_j^k \otimes 1) \cap (b_j^k \otimes u)) \\
&= \Phi_*((\bar{b}_j^k \cap b_j^k) \otimes u) = \Phi_*(\{v\} \otimes u) = \{\tau\}
\end{aligned}
$$

where $\{v\}$ is the natural generator of $H_0(X)$ determined by the inclusion of the basepoint v into X. From the proof of Lemma 4.1, $\Phi_*(\{v\} \otimes u) = \{\tau\}$. Substituting the above computation into Formula 4.3 yields $\chi_1(X; \mathbf{F})(\tau) = (\sum_{k \geqslant 0}(-1)^{k+1} m_k)\{\tau\}$. If $e_{\mathbf{F}} \neq 0$ then Lemma 4.1 implies that $\{\tau\} = 0$ and so $\chi_1(X; \mathbf{F})(\tau) = 0$. Thus the conclusion of the theorem is valid in this case. If $e_{\mathbf{F}} = 0$ then from the portion

$$H_k(X) \xrightarrow{\pi_*} H_k(B) \xrightarrow{e_{\mathbf{F}} \cap} H_{k-2}(B)$$

of the Gysin homology sequence we deduce that π_* is onto and consequently $m_k = \dim_{\mathbf{F}} H_k(B, \mathbf{F})$. Thus $\dim_{\mathbf{F}} H_*(B, \mathbf{F})$ is finite and $\sum_{k \geqslant 0}(-1)^{k+1} m_k = -\chi(B; \mathbf{F})$. $\square$

Theorem 4.2 can be used to recalculate $\chi_1(X; \mathbf{F})$ in Examples 3.8 and 3.9.

Next, we consider integer coefficients. Suppose that $S^1 \to X \xrightarrow{\pi} B$ is a smooth orientable $U(1)$-bundle over a smooth, closed, oriented manifold B. Let λ be the one dimensional subbundle of the tangent bundle of X consisting of vectors which are tangent to the circle fibers and let be ν be a complementary bundle to λ. Then $\nu \cong \pi^*(T_B)$ where T_B is the tangent bundle of B. Let $[B] \in H_n(B; \mathbf{Z})$ be the fundamental class of B where $n = \dim B$. The Euler class, $\mathrm{Eul}(\nu) \in H^n(X; \mathbf{Z})$, is given by

$$\mathrm{Eul}(\nu) = \mathrm{Eul}(\pi^*(T_B)) = \pi^*(\mathrm{Eul}(T_B)) = \chi(B)\pi^*([B]^*)$$

where $[B]^* \in H^n(B; \mathbf{Z})$ is the generator determined by the condition $[B]^*([B]) = 1$; see [MS, Corollary 11.12]. The Gysin homology sequence for $S^1 \to X \xrightarrow{\pi} B$ determines a fundamental class for X; $[X] \in H_{n+1}(X)$ is the image of $[B]$ under the homomorphism $\theta_n: H_n(B; \mathbf{Z}) \to H_{n+1}(X; \mathbf{Z})$. For any closed oriented m-dimensional manifold M, let $\mathrm{PD}_M: H^i(M) \to H_{m-i}(M)$ be the Poincaré duality isomorphism explicitly given by $\mathrm{PD}_M(x) = (-1)^{i(m-i)} x \cap [M]$ where $x \in H^i(M)$ and $[M] \in H_m(M)$ is the fundamental class ($(-1)^{i(m-i)}$ appears because of our use of Dold's sign conventions). An immediate consequence of Theorem 3.1 of [GN$_2$] is the following computation of $\chi_1(X)$ (with integer coefficients):

THEOREM 4.4. $\chi_1(X)(\tau) = -\mathrm{PD}_X(\mathrm{Eul}(\nu))$. □

THEOREM 4.5. *Under the above hypotheses,* $\chi_1(X)(\tau) = -\chi(B)\{\tau\}$.

Proof. There is a Poincaré duality isomorphism between the Gysin homology sequence and the Gysin cohomology sequence, a portion of which is shown below:

$$\begin{array}{ccccc} H_0(B;\mathbf{Z}) & \stackrel{\theta_0}{\to} & H_1(X;\mathbf{Z}) & \stackrel{\pi_*}{\to} & H_1(B;\mathbf{Z}) \\ \mathrm{PD}_B \uparrow & & \mathrm{PD}_X \uparrow & & \mathrm{PD}_B \uparrow \\ H^n(B;\mathbf{Z}) & \stackrel{\pi^*}{\to} & H^n(X;\mathbf{Z}) & \to & H^{n-1}(B;\mathbf{Z}) \end{array}$$

Let $v \in X$ be a basepoint, and let $\{\pi(v)\} \in H_0(B;\mathbf{Z})$ be the generator determined by the inclusion of $\pi(v)$ into B. From the above diagram, $\mathrm{PD}_X(\pi^*([B]^*)) = \theta_0(\{\pi(v)\})$. Also, from the proof of Lemma 4.1, $\theta_0(\{\pi(v)\}) = \{\tau\}$. Thus $\mathrm{PD}_X(\mathrm{Eul}(\nu)) = \chi(B)\{\tau\}$. Regarding the free $U(1)$-action on X as a flow, we can now invoke Theorem 4.4 to conclude that $\chi(B)\{\tau\} = -\chi_1(X)(\tau)$. □

Example 4.6. Let $\sum_g$ be a closed oriented surface of genus $g > 1$ and let L_n be a complex line bundle over $\sum_g$ with Chern number n. Let $M_{n,g}$ be the total space of the $U(1)$-bundle associated to L_n. Then $M_{n,g}$ is a closed oriented aspherical 3-manifold which fibers over $\sum_g$. The center of $\pi_1(M_{n,g})$ is the infinite cyclic group generated by τ (represented by a circle fiber); the image, $\{\tau\}$, of τ in $H_1(M_{n,g}) \cong \mathbf{Z}^{2g} \oplus \mathbf{Z}/n$ generates the $\mathbf{Z}/n$ summand. By Theorem 4.5, $\chi_1(M_{n,g}): \mathbf{Z} \to H_1(M_{n,g})$ is given by $\chi_1(M_{n,g})(\tau) = (2g-2)\{\tau\}$.

Let T^n, where $n > 1$, be the n-torus (i.e. the n-fold product of copies of $U(1)$). Let X be a closed oriented smooth manifold and let $\rho: T^n \times X \to X$ be a smooth free action of T^n. This action defines a homomorphism $\bar{\rho}: T^n \to \mathrm{Diff}(X)$ where $\mathrm{Diff}(X)$ is the diffeomorphism group of X. Let $\Gamma_\rho \subset \Gamma$ be the image of the composite:

$$\pi_1(T^n, 1) \stackrel{\bar{\rho}_\#}{\to} \pi_1(\mathrm{Diff}(X), \mathrm{id}) \to \pi_1(\mathscr{E}(X), \mathrm{id}) = \Gamma .$$

PROPOSITION 4.7. *The restriction of* $\chi_1(X): \Gamma \to H_1(X)$ *to* Γ_ρ *is the zero homomorphism.*

Proof. Since $n > 1$, if $T \subset T^n$ is a circle subgroup then $\chi(X/T) = 0$. Applying Theorem 4.5 to the bundle $T \to X \to X/T$ yields the conclusion. □

COROLLARY 4.8. *If* $n > 1$ *then* $\chi_1(T^n): \mathbf{Z}^n \to \mathbf{Z}^n$ *is zero.* □

5. A HIGHER ANALOG OF GOTTLIEB'S THEOREM

Let G be a group of type $\mathscr{F}$. Gottlieb's theorem (see Propositions 1.3 and 2.4) asserts that if $\chi(G) \neq 0$ then $Z(G)$, the center of G, is trivial. We prove an analogous theorem for $\chi_1(G; \mathbf{Q})$: if $\chi_1(G; \mathbf{Q}) \neq 0$ then the center of G is infinite cyclic provided G satisfies an extra hypothesis (explained below) related to the Bass Conjecture; see Proposition 5.2 and Theorem 5.4.

Throughout this section R will be a commutative ground ring. Let S be any associative R-algebra with unit. The Hochschild homology group $HH_0(S)$ is the R-module $S/[S, S]$ where $[S, S]$ is the R-submodule of S generated by $\{ab - ba \mid a, b \in S\}$; see §2. Recall that $K_0(S)$ is the abelian group F/A where F is the free abelian group generated by the set of all isomorphism classes $[M]$ of finitely generated projective right S-modules $M \subset \bigoplus_{i=1}^{\infty} S$ and A is the subgroup of F generated by relations of the form $[M_1 \oplus M_2] - [M_1] - [M_2]$. Since a finitely generated projective module is the image of a finitely generated free module under an idempotent homomorphism, each element of $K_0(S)$ can be represented by an idempotent matrix over S. The *Hattori-Stallings* trace $T_0: K_0(S) \to HH_0(S)$ is defined as follows. Let $A: M \to M$ be an idempotent endomorphism of a free, finitely generated right S-module M representing $x \in K_0(S)$. If $[A]$ is the matrix of A with respect to a given basis for M then $T_0(x)$ is defined to be $T_0([A]) \in HH_0(S)$.

Consider the groupring, RG, of a group G over R. Then $HH_0(RG)$ is naturally isomorphic to the free R-module generated by G_1, the set of conjugacy classes of G (see §2 for an explanation in the case $R = \mathbf{Z}$). Recall that for $g \in G$ we write $C(g) \in G_1$ for the conjugacy class of g, $HH_0(RG)_{C(g)}$ for the summand of $HH_0(RG)$ corresponding to $C(g)$ and $x_{C(g)}$ for the $C(g)$-component of $x \in HH_0(RG)$. Also write $HH_0(RG) = HH_0(RG)_{C(1)} \oplus HH_0(RG)'$ where $1 \in G$ is the identity element of G, and $HH_0(RG)'$ is the direct sum of the remaining summands. The augmentation homomorphism $\varepsilon: RG \to R$ induces a homomorphism $\varepsilon_*: HH_0(RG) \to HH_0(R) = R$.

STRONG BASS PROPERTY. We say that the group G has the *Strong Bass Property over* R, abbreviated to "SBP over R", if the image of the homomorphism $T_0: K_0(RG) \to HH_0(RG)$ lies in the $HH_0(RG)_{C(1)}$ summand.

WEAK BASS PROPERTY. We say that the group G has the *Weak Bass Property over* R, abbreviated to "WBP over R", if the composite

$$K_0(RG) \xrightarrow{T_0} HH_0(RG) \xrightarrow{\text{projection}} HH_0(RG)' \xrightarrow{\varepsilon_*} R$$

is zero.

Clearly, if G has the SBP over R then it also has WBP over R. There are well-known conjectures concerning the SBP and the WBP (see [Bass], [DV] and [St, §4.1]):

STRONG BASS CONJECTURE. Every group has the SBP over $\mathbf{Z}$.

WEAK BASS CONJECTURE. Every group has the WBP over $\mathbf{Z}$.

The corresponding conjectures are false over $\mathbf{Q}$ for a group which has nontrivial torsion; instead, one could conjecture:

STRONG BASS CONJECTURE OVER $\mathbf{Q}$. Every torsion free group has the SBP over $\mathbf{Q}$.

WEAK BASS CONJECTURE OVER $\mathbf{Q}$. Every torsion free group has the WBP over $\mathbf{Q}$.

Each element of the center of G, $Z(G)$, makes up its own conjugacy class. Given a subgroup N of $Z(G)$, let $HH_0(RG)_N = \bigoplus_{C(g) \in c(N)} HH_0(RG)_{C(g)}$ where $c(N)$ is the set of conjugacy classes in G represented by elements of N. Then $HH_0(RG) = HH_0(RG)_N \oplus HH_0(RG)'_N$ where $HH_0(RG)'_N$ is the direct sum of the summands corresponding to the conjugacy classes not in $c(N)$.

PROPERTY C. We say that the group G has *Property C over R* if there exists a non-empty subset N of $Z(G)$ such that the composite

$$K_0(RG) \xrightarrow{T_0} HH_0(RG) \xrightarrow{\text{projection}} HH_0(RG)'_N \xrightarrow{\varepsilon_*} R$$

is zero.

By taking N to be the trivial subgroup of $Z(G)$ we see that if G has the WBP over R then it also has Property C over R.

Recall that a group G is said to have finite cohomological dimension over the commutative ground ring R if there exists an integer N such that $H^k(G,M) = 0$ for all RG-modules M and for all $k > N$. Also, G is said to be of type FP_∞ over R if the trivial RG-module R has a resolution by finitely generated projective RG-modules.

The following proposition is derived from the techniques of [St, §3].

PROPOSITION 5.1. *Let R be a principal ideal domain of characteristic $p \geqslant 0$. Suppose that G is of type FP_∞ over R and has finite cohomological dimension over R. Suppose also that G has a subgroup H of finite index which has Property C over R; furthermore, if $p > 0$ assume that p does not divide $[G:H]$. If the Euler characteristic $\chi(G;R) \equiv \sum_{i\geqslant 0}(-1)^i \operatorname{rank}_R H_i(G,R)$ is non-zero modulo p then the center of G is finite.*

Proof. Since H is of finite index in G, H is also of type FP_∞ over R ([Bi, Proposition 2.5]) and has finite cohomological dimension over R ([Bi, Corollary 5.10]). Furthermore, $\chi(H;R) = [G:H]\,\chi(G;R)$ and so $\chi(H;R) \neq 0 \bmod p$.

We show that the center of H, $Z(H)$, is finite. It then follows that the center of G, $Z(G)$, is finite because there is an exact sequence $1 \to Z(G) \cap H \to Z(G) \to N_G(H)/H$, where $N_G(H)$ is the normalizer of H in G, and the groups $N_G(H)/H$ and $Z(G) \cap H \subset Z(H)$ are finite.

Since H is of type FP_∞ over R and has finite cohomological dimension over R, it follows that R has a finite resolution, $0 \to P_n \to \cdots \to P_0 \to R \to 0$, where each P_j is a finitely generated projective RH-module (combine [Bi, Proposition 4.1(b)] and [Bi, Proposition 1.5])). Let $\varepsilon: RH \to R$ be the augmentation homomorphism. Consider the commutative square:

$$
\begin{array}{ccc}
K_0(RH) & \xrightarrow{T_0} & HH_0(RH) \\
\varepsilon_* \downarrow & & \varepsilon_* \downarrow \\
K_0(R) & \xrightarrow{T_0} & HH_0(R) \cong R
\end{array}
$$

Let $\alpha = \sum_{n\geqslant 0}(-1)^n [P_n] \in K_0(RH)$. Then $\varepsilon_*(T_0(\alpha)) = T_0(\varepsilon_*(\alpha)) = \chi(H;R)\cdot 1$ where $1 \in R$ is the unity in R. The second equality is the classical Hopf trace formula over the principal ideal domain R. (Stallings ([St]) calls $T_0(\alpha) \in HH_0(RH)$ the *Euler characteristic* of the projective RH-complex P_*.) Since H is assumed to have Property C over R, there is a non-empty subset N of $Z(H)$ such that $\varepsilon_*(T_0(\alpha)) = \varepsilon_*(T_0(\alpha)_N)$.

Since $\chi(H;R) \neq 0 \mod p$, it follows that $T_0(\alpha)_{C(h)} \neq 0$ for some $h \in N \subset Z(H)$. Recall that the group $Z(H)$ acts on $HH_0(RH)$ by $(rC(h))\omega = rC(h\omega^{-1})$ where $r \in R$, $h \in H$, and $\omega \in Z(H)$. By [St, Theorem 3.4] (compare (2.3) above), $T_0(\alpha)\omega = T_0(\alpha)$ for all $\omega \in Z(H)$. Since an element of $HH_0(RH)$ is a *finite* linear combination of conjugacy classes, it follows that the condition $T_0(\alpha)_{C(h)} \neq 0$ with h as above is impossible unless $Z(H)$ is finite. □

We will be interested in groups with the property that certain of their central quotients have Property C "virtually":

PROPERTY D. Let $p \geqslant 0$ be the characteristic of R. We say that the group G has *Property D over R* if the following condition holds. Given any element τ in the center of G with the property that the extension class $e_R \in H^2(G/\langle \tau \rangle; R)$ is zero (where $\langle \tau \rangle$ is the cyclic subgroup generated by τ), there is a finite index subgroup $H \subset G/\langle \tau \rangle$ such that H has Property C over R; moreover, if $p > 0$ we require that p does not divide $[G:H]$.

The next Proposition is our "higher" analog of Gottlieb's theorem over a field of arbitrary characteristic; Theorem 5.4, below, is a more usable version over $\mathbf{Q}$.

PROPOSITION 5.2. *Let* $\mathbf{F}$ *be a field. Suppose* G *is a group of type* $\mathcal{F}$ *such that* G *has Property D over* $\mathbf{F}$. *If* $\chi_1(G;\mathbf{F}) \neq 0$, *then the center of* G *is infinite cyclic.*

Proof. Let τ be any element in $Z(G)$, the center of G, such that $\chi_1(G;\mathbf{F})(\tau) \neq 0$. Since G is necessarily torsion free, the group $T = \langle \tau \rangle$ is infinite cyclic. By [Bi, Proposition 2.7] G/T is of type FP_∞ over $\mathbf{Z}$ (and hence over any commutative ring). Since T is central, the Serre fibration $S^1 \simeq K(T,1) \to K(G,1) \to K(G/T,1)$ is orientable. By Theorem 4.2, $e_{\mathbf{F}} = 0 \in H^2(G/T;\mathbf{F})$, and $\chi(G/T;\mathbf{F})$ exists and is non-zero mod p where $p \geqslant 0$ is the characteristic of $\mathbf{F}$. Consider the following portion of the cohomology Gysin sequence of the fibration $S^1 \to K(G,1) \to K(G/T,1)$, with coefficients in an arbitrary $\mathbf{F}G/T$-module M:

$$H^{i-2}(G/T;M) \xrightarrow{\cup e_{\mathbf{F}}} H^i(G/T;M) \to H^i(G;M)\,.$$

Since $e_{\mathbf{F}} = 0$, $H^i(G/T;M) \to H^i(G;M)$ is injective and so $H^i(G/T,M) = 0$ for $i > \dim X$ where X is a finite complex homotopy equivalent to $K(G,1)$. In particular, Proposition 5.1 applies to G/T and so the center of G/T is finite. Since the image of $Z(G)$ in G/T is central, it follows that $Z(G)$ is an extension of T by a finite group. Thus $Z(G)$ is infinite cyclic since G is torsion free. $\square$

Property D may be hard to verify for an arbitrary coefficient ring R. However, when $R = \mathbf{Q}$ we have:

PROPOSITION 5.3. *Let G be a finitely generated group which has the WBP over $\mathbf{Q}$. Then G has Property D over $\mathbf{Q}$.*

Proof. Suppose $\tau \in Z(G)$ is such that the extension class $e_{\mathbf{Q}} \in H^2(G/T; \mathbf{Q})$ is zero where T is the cyclic subgroup of G generated by τ. Consider the following portion of the long exact sequence in cohomology associated to the short exact sequence of coefficients, $0 \to \mathbf{Z} \xrightarrow{j} \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$:

$$H^1(G/T; \mathbf{Q}/\mathbf{Z}) \xrightarrow{\delta} H^2(G/T; \mathbf{Z}) \xrightarrow{j_*} H^2(G/T; \mathbf{Q}) .$$

By exactness, $j_*(e_{\mathbf{Z}}) = e_{\mathbf{Q}} = 0$ implies $e_{\mathbf{Z}} = \delta(u)$ for some $u \in H^1(G/T, \mathbf{Q}/\mathbf{Z})$. Let $H = \ker(u)$ where we regard u as an element of $\operatorname{Hom}(G/T, \mathbf{Q}/\mathbf{Z}) \cong H^1(G/T, \mathbf{Q}/\mathbf{Z})$. Since G is finitely generated, $H \overset{i}{\hookrightarrow} G/T$ is of finite index. Let $H' = \pi^{-1}(H)$ where $\pi : G \to G/T$ is the quotient homomorphism. Then H' is isomorphic to $H \times T$ because $i^*(e_{\mathbf{Z}}) = 0$. In particular, H is isomorphic to a subgroup of G. Let $\mu : H \to G$ be a monomorphism. The commutative diagram

$$\begin{array}{ccc} K_0(\mathbf{Q}H) & \xrightarrow{T_0} & HH_0(\mathbf{Q}H) \\ \mu_* \downarrow & & \mu_* \downarrow \\ K_0(\mathbf{Q}G) & \xrightarrow{T_0} & HH_0(\mathbf{Q}G) \end{array}$$

and the observation that $\mu_*(HH_0(\mathbf{Q}H))_{C(1)} \subset HH_0(\mathbf{Q}G)_{C(1)}$ and $\mu_*(HH_0(\mathbf{Q}H)') \subset HH_0(\mathbf{Q}G)'$ imply that H has the WBP over $\mathbf{Q}$ (and thus Property C over $\mathbf{Q}$). $\square$

Combining Propositions 5.2 and 5.3 we get:

THEOREM 5.4. *Suppose that G is a group of type $\mathcal{F}$ and has the WBP over $\mathbf{Q}$. If $\chi_1(G; \mathbf{Q}) \neq 0$, then the center of G is infinite cyclic.* $\square$

Groups of type $\mathcal{F}$ are a very special class of torsion free groups; one would hope that all groups of type $\mathcal{F}$ have the WBP over $\mathbf{Q}$. There are special classes of groups of type $\mathcal{F}$ which are known to have the WBP over $\mathbf{Q}$. We recall two such classes.

A group G is a *linear group* if it is a subgroup of $GL(n, \mathbf{K})$ where $\mathbf{K}$ is a field of characteristic zero. Bass [Bass, Theorem 9.6] proved that a torsion free linear group has the SBP over $\mathbf{C}$ (and thus has the WBP over $\mathbf{Q}$); also see [Eck].

COROLLARY 5.5. *Suppose G is a linear group of type $\mathscr{F}$. If $\chi_1(G; \mathbf{Q}) \neq 0$, then the center of G is infinite cyclic.* □

Eckmann [Eck] proved that a group of cohomological dimension 2 over $\mathbf{Q}$ has the SBP over $\mathbf{Q}$. Consequently:

COROLLARY 5.6. *Suppose G is of type $\mathscr{F}$ and has cohomological dimension 2 over $\mathbf{Q}$. If $\chi_1(G; \mathbf{Q}) \neq 0$, then the center of G is infinite cyclic.* □

There is a sense in which we can say that $\chi_1(G; \mathbf{Q})$ is an integer. Denote the composite homomorphism $Z(G) \hookrightarrow G \xrightarrow{A} H_1(G; \mathbf{Z}) \to H_1(G; \mathbf{Q})$ by $A_{\mathbf{Q}}: Z(G) \to H_1(G; \mathbf{Q})$.

THEOREM 5.7. *Let G be a group of type $\mathscr{F}$ which has the WBP over $\mathbf{Q}$. Then there exists an integer n_G (depending only on G) such that $\chi_1(G; \mathbf{Q}) = n_G A_{\mathbf{Q}}$.*

Proof. If $\chi_1(G; \mathbf{Q}) = 0$ take $n_G = 0$. If $\chi_1(G; \mathbf{Q}) \neq 0$ then by Theorem 5.4 the center of G is infinite cyclic. Let $\tau \in Z(G)$ generate $Z(G)$. Since $\chi_1(G; \mathbf{Q}) \neq 0$ we have $\chi_1(G; \mathbf{Q})(\tau) \neq 0$. By Theorem 4.2, $\chi_1(G; \mathbf{Q})(\tau) = -\chi(G/\langle\tau\rangle; \mathbf{Q})\{\tau\}$. Then for any integer r: $\chi_1(G; \mathbf{Q})(\tau^r) = r\chi_1(G; \mathbf{Q})(\tau) = -r\chi(G/\langle\tau\rangle; \mathbf{Q})\{\tau\} = -\chi(G/\langle\tau\rangle; \mathbf{Q})A_{\mathbf{Q}}(\tau^r)$. Thus $\chi_1(G; \mathbf{Q}) = n_G A_{\mathbf{Q}}$ with $n_G = -\chi(G/\langle\tau\rangle; \mathbf{Q})$. □

Remarks.

1. All integers occur as n_G for some G. Given $n \in \mathbf{Z}$, there is a group H of type $\mathscr{F}$ with $\chi(H) = -n$ (e.g. take H to be an appropriate Cartesian product of free groups). Let $G = H \times T$ where T is infinite cyclic. Clearly, $\chi(G/\langle\tau\rangle; \mathbf{Q}) = \chi(H)$ where τ is a generator of $(1) \times T \subset G$ and so $\chi_1(G; \mathbf{Q}) = nA_{\mathbf{Q}}$ (alternatively, see Example 6.15).

2. Theorem 5.7 remains true without the hypothesis that G has the WBP over $\mathbf{Q}$ although the proof is considerably more lengthy. To prove this strengthened result, one shows that for *any* group G of type $\mathscr{F}$:

(a) The restriction of $\chi_1(G; \mathbf{Q})$ to $Z(G) \cap [G, G]$ is zero.

(b) If $\chi_1(G; \mathbf{Q}) \neq 0$ then $\dim_{\mathbf{Q}} A_{\mathbf{Q}}(Z(G)) = 1$.

The desired conclusion follows easily from (a), (b) and Theorem 4.2.

Theorem 5.7 raises the question: For what groups G of type $\mathscr{F}$ is $\chi_1(G, \mathbf{Q}) \neq 0$? We give a necessary condition. Recall that a group H has type $\mathscr{F}\mathscr{D}$ if there is a finitely dominated $K(H, 1)$ (i.e. $K(H, 1)$ is a homotopy retract of a finite complex).

PROPOSITION 5.8. *If* $\chi_1(G, \mathbf{Q}) \neq 0$ *then* G *is isomorphic to a semidirect product* $\langle H, t \mid tht^{-1} = \theta(h)$ *for all* $h \in H \rangle$ *where* H *has type* $\mathscr{F}\mathscr{D}$.

Proof. Let $\tau \in Z(G)$ be such that $\chi_1(G, \mathbf{Q})(\tau) \neq 0$. By Theorem 4.2, it follows that $\{\tau\} \in H_1(G) \equiv G_{\mathrm{ab}}$ is of infinite order. Thus there is an epimorphism $p: G \to \mathbf{Z}$ with $p(\tau) = n$ for some $n > 0$. Let $H = \ker(p)$. Since $\tau \in Z(G)$, $p^{-1}(n\mathbf{Z}) \cong H \times \mathbf{Z}$ and has finite index in G. Thus $H \times \mathbf{Z}$ has type $\mathscr{F}$ and so H has type $\mathscr{F}\mathscr{D}$. $\square$

Thus it is worthwhile to compute $\chi_1(G, \mathbf{Q})$ in terms of such a semidirect product structure. The geometric problem underlying this is the study of $\chi_1(X)$ where X is a mapping torus. We study this next, returning to the group theoretic case in §7.

6. MAPPING TORI

In this section, we consider $\chi_1(X)$ and $\tilde{\chi}_1(X)$ when X is the mapping torus of a map $f: Z \to Z$. The main results are Theorems 6.3, 6.13, 6.14, 6.16 and Corollary 6.18. Applications to the aspherical case will be given in §7.

Suppose Z is a path connected space and has a basepoint $v \in Z$. Given a continuous map $f: Z \to Z$, its *mapping torus*, denoted by $T(Z, f)$, is the space obtained from $Z \times [0, 1]$ by identifying $(z, 1)$ with $(f(z), 0)$ for each $z \in Z$. The image of $(z, u) \in Z \times [0, 1]$ in $T(Z, f)$ will be denoted by $[z, u]$. Choose a basepath σ from v to $f(v)$ and let $\theta: H \to H$ be the self homomorphism of $H \equiv \pi_1(Z, v)$ determined by f and σ.

Let $X = T(Z, f)$. Choose $w = [v, 0]$ as a basepoint for X and let $G = \pi_1(X, w)$. There is a canonical map of X to the standard circle S^1 (realized as complex numbers of unit modulus) given by: $p_f: X \to S^1$, $p_f([z, s]) = e^{2\pi i s}$. Let $i: Z \hookrightarrow X$ be the inclusion $z \mapsto [z, 0]$.

Recall that $\Gamma = \pi_1(\mathscr{E}(X), \mathrm{id})$. Let $\Gamma_{S^1} = \pi_1(\mathscr{E}(S^1), \mathrm{id})$. Let $\bar{\gamma}: I \to X$ be the path $\bar{\gamma}(u) = [v, u]$ and let $\gamma_0: I \to X$ be the path $\gamma_0 = \bar{\gamma}(i \circ \sigma)^{-1}$. Define a continuous map $P: X^X \to (S^1)^{S^1}$ by $P(g)(e^{2\pi i u}) = p_f(g(\gamma_0(u)))$. Then P induces a homomorphism $P_*: \Gamma \to \Gamma_{S^1}$. We define an identification $\Gamma_{S^1} \xrightarrow{\cong} \mathbf{Z}$ by sending the generator $[s \mapsto (e^{2\pi i u} \mapsto e^{2\pi i(u+s)})] \in \Gamma_{S^1}$ to $1 \in \mathbf{Z}$. The *rotation degree* of $\gamma \in \Gamma$ is the integer $P_*(\gamma)$.

We now describe some useful homotopies of X.

For a non-negative integer k, the *k-th tumble* is the homotopy which "rolls the mapping torus through an angle of $2\pi k$"; explicitly, this homotopy, denoted by $R_k: X \times [0, 1] \to X$, is given by the formula $R_k([z, u], s) = [f^{[ks+u]}(z), (ks + u) \bmod 1]$ where $[ks + u]$ is the integer part of $ks + u$.

Whenever a map $g: Z \to Z$ commutes with f (i.e. $fg = gf$), there is an induced "level" map $\hat{g}: X \to X$ given by $\hat{g}([z, u]) = [g(z), u]$; for example, the k-th tumble, R_k, is a homotopy from id_X to $\hat{f}^k$. We need a more general procedure (see Proposition 6.2 below) for extending homotopies of Z to homotopies of X.

A homotopy $N: Z \times I \to Z$ *eventually commutes* with f if there exists an integer $m \geqslant 0$ and a homotopy $J: Z \times I \times I \to Z$ with $J(z, u, 0) = f^m \circ N(f(z), u)$, $J(z, u, 1) = f^{m+1} \circ N(z, u)$, $J(z, 0, s) = f^m \circ N(f(z), 0)$, $J(z, 1, s) = f^m \circ N(f(z), 1)$. Thus J makes the following diagram commute up to homotopy rel $Z \times \{0, 1\} \times I$:

$$\text{(6.1)} \qquad \begin{array}{ccccc} Z \times I & \xrightarrow{f \times \mathrm{id}} & Z \times I & \xrightarrow{N} & Z \\ \downarrow N & & & & \downarrow f^m \\ Z & \xrightarrow{f^{m+1}} & Z & = & Z \end{array}$$

This implies $f^m \circ N_i \circ f = f^{m+1} \circ N_i$ for $i = 0, 1$; in our applications, N_0 and N_1 will be iterates of f.

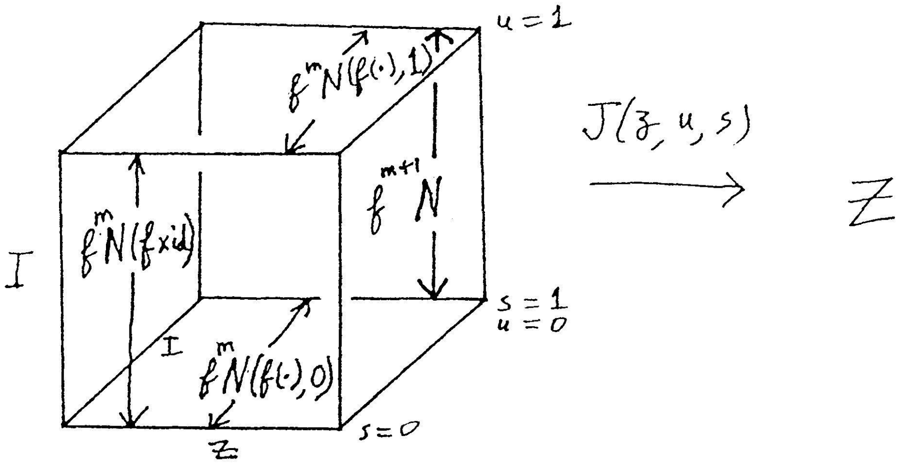

FIGURE 1

Define $L'_{(N,J,m)}: X \times I \to X$ (abbreviated to L') by the formula:

$$L'([z,u],s) = \begin{cases} [f^m \circ N(z,s), 2u] & \text{if } 0 \leqslant u \leqslant \frac{1}{2} \\ [J(z,s,2-2u), 0] & \text{if } \frac{1}{2} \leqslant u \leqslant 1 \,. \end{cases}$$

and define $K: X \times I \to X$ by:

$$K([z,u],s) = \begin{cases} [z, u(1+s)] & \text{if } 0 \leqslant u \leqslant \frac{1}{2} \\ [z, s(1-u)+u] & \text{if } \frac{1}{2} \leqslant u \leqslant 1 \,. \end{cases}$$

(K is a "linear" homotopy from id_X to a map which sends the points $[z,u]$, $\frac{1}{2} \leqslant u \leqslant 1$, to $[z,1] \equiv [f(z),0]$.)

Observe that $L'(\,\cdot\,,0) = K(\,\cdot\,,1) \circ (\widehat{f^m \circ N_0})$ and $L'(\,\cdot\,,1) = K(\,\cdot\,,1) \circ (\widehat{f^m \circ N_1})$. Thus, for N, J and m as above, we have:

PROPOSITION 6.2. *The concatenation*

$$L_{(N,J,m)} \equiv K \circ (\widehat{f^m \circ N_0} \times \mathrm{id}) \bigstar L'_{(N,J,m)} \bigstar (K \circ (\widehat{f^m \circ N_1} \times \mathrm{id}))^{-1}$$

is a homotopy from $\widehat{f^m \circ N_0}$ *to* $\widehat{f^m \circ N_1}$. $\square$

(For a homotopy Q, Q^{-1} means the homotopy $Q^{-1}(x,s) = Q(x,1-s)$.)

Next, we will build special elements of Γ. The map $f: Z \to Z$ is a *periodic homotopy idempotent* if there exists $r \geqslant 0$ and $q > 0$ such that f^r is homotopic to f^{r+q}; it is not assumed that this can be achieved by a basepoint preserving homotopy. If for some $r \geqslant 0$ and $q > 0$ there is a homotopy $N: f^r \simeq f^{r+q}$ for which there exist J and $m \geqslant 0$ making Diagram 6.1, commute up to homotopy rel $Z \times \{0,1\} \times I$, then we say that f is *eventually coherent*. In this case, Proposition 6.2 gives a homotopy $L_{(N,J,m)}: \hat{f}^{r+m} \simeq \hat{f}^{r+q+m}$. The concatenation $S \equiv S_{(r,N,J,m)} \equiv R_{r+q+m} \bigstar L^{-1}_{(N,J,m)} \bigstar R^{-1}_{r+m}$ is a homotopy from id_X to id_X whose rotation degree is q. Given f, the least $q > 0$ for which there exist r, N, J and m as above (assuming that they exist at all) is the *period* of f. Then r and m may be chosen as large as desired.

These conditions on a map f which give rise to an element $[S] \in \Gamma$ having positive rotation degree, are not arbitrary. Rather, they are the general case:

THEOREM 6.3. *Let* $f: Z \to Z$ *be a map for which the rotation degree homomorphism* $P_*: \Gamma \to \mathbf{Z}$ *is non-zero. Let* q *be the least positive element of* $P_*(\Gamma)$. *Then* f *is an eventually coherent periodic homotopy idempotent of period* q.

Before proving this, we set up notation for points of the infinite mapping telescope of f, i.e. the infinite cyclic cover of X whose fundamental group is H. This space, denoted by $\bar{X}$, is the quotient of the disjoint union $\coprod_{n \in \mathbf{Z}} Z \times \{n\} \times [0, 1]$ obtained by identifying $(z, n, 1)$ with $(f(z), n + 1, 0)$ for all n. The image of (z, n, u) in $\bar{X}$ will be denoted by $[z, n, u]$. The covering projection $\bar{X} \to X$ is given by $[z, n, u] \mapsto [z, (n + u) \bmod 1]$. The space $\bar{X}$ is a "two-ended union" of mapping cylinders: we write $M(f)_n$ for the subset of points $[z, n, u]$ such that $0 \leqslant u \leqslant 1$, and Z_n for the subset of points $[z, n, 0]$.

Proof of 6.3. Let $F^\gamma : \mathrm{id}_X \simeq \mathrm{id}_X$ represent $\gamma \in \Gamma$ of rotation degree q, and let $\bar{F}^\gamma : \bar{X} \times I \to \bar{X}$ be the basepoint preserving lift of F^γ. The map $\bar{F}^\gamma$ is a homotopy between $\mathrm{id}_{\bar{X}}$ and $\mathscr{T}^q$, where $\mathscr{T}([z, n, u]) = [z, n + 1, u]$ is "translation by 1". Let $i_n : Z \to \bar{X}$ be the "inclusion" of Z as Z_n, i.e. $i_n(z) = [z, n, 0]$. The composition $Z \times I \xrightarrow{i_0 \times \mathrm{id}} \bar{X} \times I \xrightarrow{\bar{F}^\gamma} \bar{X}$ gives a homotopy between i_0 and i_q. The formula $(z, s) \mapsto [f^{[sq]}(z), [sq], sq - [sq])$ gives a homotopy between i_0 and $i_q \circ f^q$. Combining the two, we get a homotopy $\Phi : i_q \simeq i_q \circ f^q$. The track of $\Phi : Z \times I \to \bar{X}$ lies in $\cup_{n=r'}^{r+q-1} M(f)_n$ for suitable integers $r' \leqslant 0 < q \leqslant r + q$. Form a homotopy $\Psi : Z \times I \to \bar{X}$, $\Psi : i_{r+q} \circ f^r \simeq i_{r+q} \circ f^{r+q}$, whose entire image lies in Z_{r+q}, by "pushing" the track of Φ along the mapping telescope into Z_{r+q}; explicitly, if $\Phi(z, s) = [z', n', u']$ then $\Psi(z, s) = [f^{r+q-n'}(z'), r + q, 0]$. Identifying Z_{r+q} with Z, we get a homotopy between f^r and f^{r+q}.

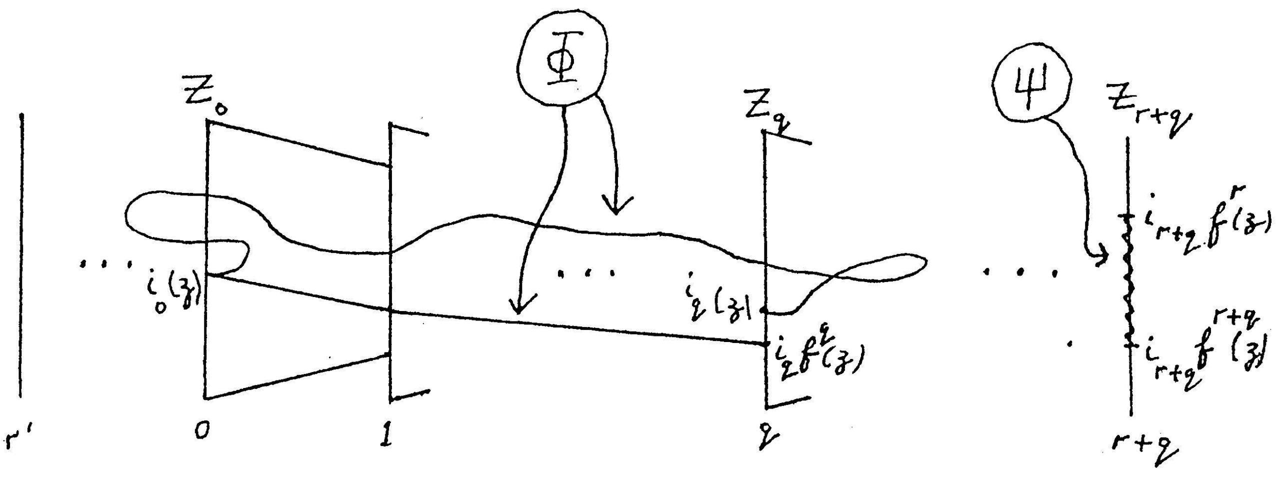

FIGURE 2

It remains to prove eventual coherence. Since $\bar{F}^\gamma$ is $\mathbf{Z}$-equivariant (with respect to the $\mathbf{Z}$-action generated by $\mathscr{T}$), there is a $\mathbf{Z}$-equivariant homotopy $\bar{\Psi} : \bar{X} \times I \to \bar{X}$ such that $\Psi = \bar{\Psi} \circ (i_0 \times \mathrm{id})$.

Consider the diagram:

$$\begin{array}{ccccccc}
Z \times I & \overset{f \times \mathrm{id}}{\leftarrow} & Z \times I & \overset{\Psi}{\rightarrow} & \bar{X} & \overset{\bar{\bar{f}}}{\rightarrow} & \bar{X} \\
\downarrow i_1 \times \mathrm{id} & & \downarrow i_0 \times \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow \mathscr{E} \\
\bar{X} \times I & = & \bar{X} \times I & \overset{\bar{\Psi}}{\rightarrow} & \bar{X} & = & \bar{X} \\
 & & \uparrow i_1 \times \mathrm{id} & & \uparrow \mathscr{E} & & \\
 & & Z \times I & \overset{\Psi}{\rightarrow} & \bar{X} & &
\end{array}$$

The two middle squares commute. The upper right square commutes up to the homotopy given by:

$$([z, n, u], s) \mapsto \begin{cases} [z, n, u + s] & \text{if } 0 \leqslant s \leqslant 1 - u \\ [f(z), n + 1, u + s - 1) & \text{if } 1 - u \leqslant s \leqslant 1 \,. \end{cases}$$

There is a corresponding homotopy for the upper left square. Thus, the diagram

$$\begin{array}{ccc}
Z \times I & \overset{\Psi}{\rightarrow} & \bar{X} \\
\downarrow f \times \mathrm{id} & & \downarrow \bar{\hat{f}} \\
Z \times I & \overset{\Psi}{\rightarrow} & \bar{X}
\end{array}$$

commutes up to a homotopy $J' : \Psi \circ (f \times \mathrm{id}) \simeq \bar{\hat{f}} \circ \Psi$ which has the property that, for $i = 0$ or 1, the restriction $J'| : Z \times \{i\} \times I \to \bar{X}$ is homotopic rel $Z \times \{i\} \times \{0, 1\}$ to a constant homotopy. Thus, adjusting J', we obtain a homotopy $J'' : Z \times I \times I \to \bar{X}$ rel $Z \times \{0, 1\} \times I$ between $\Psi \circ (f \times \mathrm{id})$ and $\bar{\hat{f}} \circ \Psi$. The argument is finished by "pushing" the track of J'' along the mapping telescope into Z_{r+q+m} where $r + q + m$ is sufficiently large: the details are similar to the construction of Ψ from Φ. We then obtain a homotopy commutative diagram similar to (6.1), showing that f is as claimed. □

Remark. We do not know if every periodic homotopy idempotent $f : Z \to Z$ is eventually coherent. The special case of interest for group theory is the case where Z is aspherical and f is a homotopy equivalence so that we are essentially concerned with an element of the outer automorphism group of $\pi_1(Z)$. A consequence of Proposition 7.3 is that f is indeed eventually coherent in this situation. In the more general case where f is homotopy equivalence but Z is not necessarily aspherical, the obstruction theory of [C] is relevant; see [GN_4].

If (r, N, J, m) are, as above, the data for an eventually coherent periodic homotopy idempotent of period q, we can form $(r, N \bigstar (f^q \circ N), J \bigstar (f^q \circ J), m)$. Here, $N^{(2)} \equiv N \bigstar (f^q \circ N) : f^r \simeq f^{r+2q}$, and the concatenation $J^{(2)} \equiv J \bigstar (f^q \circ J)$ takes place in the first I-factor, so that it coincides (after suitable reparametrization) with J on $Z \times \left[0, \frac{1}{2}\right] \times I$ and with $f^q \circ J$ on $Z \times \left[\frac{1}{2}, 1\right] \times I$. One verifies that $(r, N^{(2)}, J^{(2)}, m)$ make Diagram 6.1 commute, hence one has, as above, $S_{(r, N^{(2)}, J^{(2)}, m)} \equiv R_{r+2q+m} \bigstar L^{-1}_{(N^{(2)}, J^{(2)}, m)} \bigstar R^{-1}_{r+m}$, a homotopy from id_X to itself whose rotation degree is $2q$. Iterating this procedure one gets, for any positive integer ν, $S_{(r, N^{(\nu)}, J^{(\nu)}, m)} \equiv R_{r+\nu q+m} \bigstar L^{-1}_{(N^{(\nu)}, J^{(\nu)}, m)} \bigstar R^{-1}_{r+m}$, a homotopy from id_X to itself of rotation degree νq.

PROPOSITION 6.4. *With f and $q > 0$ as in Theorem 6.3, and ν a positive integer, let $\gamma \in \Gamma$ have rotation degree νq. Let (r, N, J, m) be data exhibiting f as an eventually coherent periodic homotopy idempotent of period q. Then there exists $\delta \in \Gamma$ of rotation degree 0 such that $\gamma = \delta[S_{(r, N^{(\nu)}, J^{(\nu)}, m)}]$.*

Proof. Take δ to be $\gamma[S_{(r, N^{(\nu)}, J^{(\nu)}, m)}]^{-1}$. □

Elements of Γ having rotation degree 0 can be "regularized". Let $F^\delta : \mathrm{id}_X \simeq \mathrm{id}_X$ represent such a δ. The basepoint preserving lift is $\bar{F}^\delta : \bar{X} \times I \to \bar{X}$, a homotopy from $\mathrm{id}_{\bar{X}}$ to $\mathrm{id}_{\bar{X}}$. As in the proof of Theorem 6.3, there is an integer $l \geqslant 0$ such that the track, under $\bar{F}^\delta$, of every point $[z, n, u] \in \bar{X}$ can be "pushed" equivariantly into $\{[y, n+l, u] \mid y \in Z\}$. Thus, by an obvious further adjustment, we have:

PROPOSITION 6.5. *If δ has rotation degree 0, then for any sufficiently large l (dependent on δ), F^δ is homotopic rel $X \times \{0, 1\}$ to a homotopy of the form $R_l \bigstar L^{-1}_{(N, J, 0)} \bigstar R_l^{-1}$ where $N : f^l \simeq f^l$ is constructed from F^δ as in the proof of Theorem 6.3.* □

We now prepare to compute the derivation $\tilde{\mathrm{X}}_1(X) : \Gamma \to HH_1(\mathbf{Z}G)$.

In the remainder of this section we assume that Z is a finite CW complex, that the map f is cellular and that the basepath σ is cellular. Then $X = T(Z, f)$ inherits a natural CW structure. We will also assume that f is a π_1-*equivalence*; i.e. the induced map $f_\# : \pi_1(Z, v) \to \pi_1(Z, f(v))$ is an isomorphism. Thus $\theta : H \to H$, defined above, is an automorphism. Then the group G is a semidirect product of H with $T \equiv \pi_1(S^1, 1)$; there is an exact sequence: $H \rightarrowtail G \twoheadrightarrow T$ where $H \rightarrowtail G$ is induced by the inclusion $i : Z \hookrightarrow X$

and $G \twoheadrightarrow T$ is induced by p_f. We write $t = [\gamma_0]^{-1} \in G$, projecting to a generator of T, so that $\theta: H \to H$ is given by $h \mapsto tht^{-1}$. We make this choice because we deal with right modules; here and in [GN$_2$] we prefer "t" rather than "t^{-1}" to appear in our matrices.

Since $\theta: H \to H$ is an isomorphism, the universal cover, $\tilde{X}$, of $X = T(Z, f)$ can be thought of as the mapping telescope of $\tilde{f}: \tilde{Z} \to \tilde{Z}$. Then we have the following model, denoted by $C_*(\tilde{X})$, for the cellular chain complex of $\tilde{X}$. Let $(C_*(\tilde{Z}), {}_Z\tilde{\partial})$ be the cellular chain complex of $\tilde{Z}$. Define $C_*(\tilde{X})$ by

$$C_n(\tilde{X}) = (C_{n-1}(\tilde{Z}) \oplus C_n(\tilde{Z})) \otimes_{\mathbf{Z}} \mathbf{Z}[t, t^{-1}]$$

where the right action of G on $C_n(\tilde{X})$ is given as follows: if $ht^j \in G$ and $a \otimes t^i \in C_n(\tilde{X})$ then $(a \otimes t^i)ht^j \equiv a\theta^i(h) \otimes t^{i+j}$. A choice of oriented lifts of the $(n-1)$-cells and the n-cells of Z determines a finite $\mathbf{Z}G$ basis for the right $\mathbf{Z}G$-module $C_n(\tilde{X})$. The matrix of the boundary operator ${}_X\tilde{\partial}_{n+1}: C_{n+1}(\tilde{X}) \to C_n(\tilde{X})$ with respect to the given $\mathbf{Z}G$ bases is:

$$\begin{bmatrix} [{}_Z\tilde{\partial}_n] & 0 \\ (-1)^{n+1}(I - [\tilde{f}_n]t) & [{}_Z\tilde{\partial}_{n+1}] \end{bmatrix}$$

where $[{}_Z\tilde{\partial}_n]$ is the matrix of ${}_Z\tilde{\partial}_n$, $[\tilde{f}_n]$ is the matrix of $\tilde{f}_n$ and I is an identity matrix of the same size as $[\tilde{f}_n]$. For background on the following calculations, the reader is referred to [GN$_2$, §4]. See also the Sign Convention in §1.

Let $(\tilde{\mathscr{R}}_k)_n: C_n(\tilde{X}) \to C_{n+1}(\tilde{X})$ be the chain homotopy defined by the k-th tumble R_k. The matrix for $(\tilde{\mathscr{R}}_k)_n$ is:

$$\begin{bmatrix} 0 & (-1)^{n+1} \sum_{i=0}^{k-1} ([\tilde{f}_n]t)^i \\ 0 & 0 \end{bmatrix}.$$

Thus we have:

PROPOSITION 6.6. $\operatorname{trace}(\tilde{\partial} \otimes \tilde{\mathscr{R}}_k)$ *is the Hochschild* 1*-chain*

$$\sum_{n \geqslant 0} (-1)^n \operatorname{trace}(([\tilde{f}_n]t) \otimes \sum_{i=0}^{k-1} ([\tilde{f}_n]t)^i).$$

Proof. The identity $d(1 \otimes 1 \otimes g) = 1 \otimes g$ implies that terms of the form $\operatorname{trace}(I \otimes M)$ are boundaries and can therefore be ignored. □

Next, suppose f is an eventually coherent periodic homotopy idempotent. As above, we have $r \geqslant 0$, $N: f^r \simeq f^{r+q}$, $m \geqslant 0$, and $J: Z \times I \times I \to Z$;

and $L_{(N,J,m)}$ is a homotopy from f^{r+m} to $\tilde{f}^{r+q+m}$. By Proposition 6.2, $L_{(N,J,m)}$ is the concatenation of three homotopies: the first and third of these have zero matrices at the chain homotopy level, and the second, which is $L'_{(N,J,m)}$, is easily seen to give a chain homotopy whose block for $C_n(\tilde{X}) \to C_{n+1}(\tilde{X})$ is

$$\begin{bmatrix} [\tilde{f}^m_n]\,[\tilde{\mathscr{N}}_{n-1}] & 0 \\ W & [\tilde{f}^m_{n+1}]\,[\tilde{\mathscr{N}}_n] \end{bmatrix}.$$

Here, $\tilde{\mathscr{N}}: C_*(\tilde{Z}) \to C_{*+1}(\tilde{Z})$ is the chain homotopy defined by N, and W is a matrix whose exact nature need not concern us. Because of our sign conventions, and the fact that the upper right block is zero we get:

PROPOSITION 6.7. *Let* $\tilde{\mathscr{L}}_{(N,J,m)}: C_*(\tilde{X}) \to C_{*+1}(\tilde{X})$ *be the chain homotopy determined by* $L_{(N,J,m)}$. *Then* $\operatorname{trace}(\tilde{\partial} \otimes \tilde{\mathscr{L}}_{(N,J,m)}) = 0$. $\square$

Now, let $\delta \in \Gamma$ have rotation degree 0. Then $\eta_\#(\delta)$ lies in $H \subset G$ (where η is defined in §1). By Proposition 6.5, we may take $F^\delta = R_l \star L^{-1}_{(N,J,0)} \star R_l^{-1}$ for any sufficiently large l. Under the homotopy $F^\delta: \mathrm{id}_X \simeq \mathrm{id}_X$, the basepoint traverses a loop representing $\eta_\#(\delta)$. Let $\tilde{D}^\delta$ be the chain homotopy defined by $\tilde{F}^\delta$. We rewrite $\eta_\#(\delta) = t^{-l}(t^l\eta_\#(\delta)t^{-l})t^l$. At the matrix level, we then have:

$$\tilde{D}^\delta = \tilde{\mathscr{R}}_l - \tilde{\mathscr{L}}_{(N,J,0)}(t^l\eta_\#(\delta)t^{-l})t^l - \tilde{\mathscr{R}}_l t^{-l}(t^l\eta_\#(\delta)^{-1}).$$

Here, we have used the fact that the matrix of a chain homotopy for a concatenation $A \star B$ is $\mathscr{A} + \mathscr{B}g^{-1}$ where $\mathscr{A}$ and $\mathscr{B}$ are the matrices of A and B and $g \in G$ is the element represented by $A(\text{basepoint} \times I)$, and the matrix for A^{-1} is $-\mathscr{A}g$. In what follows, recall the right action of Γ on Hochschild chains and homology described in §2. Using Proposition 6.7 we get:

COROLLARY 6.8. *If* $\delta \in \Gamma$ *has rotation degree* 0, *then* $\tilde{\mathrm{X}}_1(X)(\delta)$ *is represented by the Hochschild cycle*

$$\operatorname{trace}(\tilde{\partial} \otimes \tilde{D}^\delta) = \operatorname{trace}(\tilde{\partial} \otimes \tilde{\mathscr{R}}_l)(1 - \delta^{-1})$$

for any sufficiently large l *(dependent on* δ*).* $\square$

Now we return to the situation discussed in Theorem 6.3 and Proposition 6.4. We have $\gamma \in \Gamma$ of rotation degree νq. By Proposition 6.4, $\gamma = \delta[S^{(\nu)}]$ where δ is represented by F^δ, and, for suitably large r and m

(depending only on f), $S^{(\nu)} = R_{r+\nu q+m} \star L^{-1}_{(N^{(\nu)}, J^{(\nu)}, m)} \star R^{-1}_{r+m}$. Under F^{γ}, the basepoint traces out a loop representing

$$\eta_{\#}(\delta)\, t^{-r-\nu q-m}\left(t^{r+\nu q+m}\eta_{\#}(\delta)^{-1}\eta_{\#}(\gamma)\, t^{-r-m}\right) t^{r+m} = \eta_{\#}(\gamma)\,.$$

Here, the four factors correspond to the four parts of the concatenation. Thus

$$\begin{aligned}\operatorname{trace}(\tilde{\partial} \otimes \tilde{D}^{\gamma}) &= \operatorname{trace}(\tilde{\partial} \otimes \tilde{D}^{\delta}) + \operatorname{trace}(\tilde{\partial} \otimes \tilde{\mathscr{R}}_{r+\nu q+m}\eta_{\#}(\delta)^{-1}) \\ &\quad - \operatorname{trace}(\tilde{\partial} \otimes \tilde{\mathscr{L}}_{(N,J,m)}\, t^{r+\nu q+m}\eta_{\#}(\delta)^{-1}) \\ &\quad - \operatorname{trace}(\tilde{\partial} \otimes \tilde{\mathscr{R}}_{r+m}(\gamma)^{-1})\,.\end{aligned}$$

Using Proposition 6.7 and Corollary 6.8 and the right Γ-action described in Proposition 2.6, this becomes:

$$\begin{aligned}\operatorname{trace}(\tilde{\partial} \otimes \tilde{D}^{\gamma}) = \operatorname{trace}(\tilde{\partial} \otimes \tilde{\mathscr{R}}_{l})(1-\delta^{-1}) + \operatorname{trace}(\tilde{\partial} \otimes \tilde{\mathscr{R}}_{r+\nu q+m})\delta^{-1} \\ - \operatorname{trace}(\tilde{\partial} \otimes \tilde{\mathscr{R}}_{r+m})\gamma^{-1}\,.\end{aligned}$$

In particular, if we enlarge l or $r+m$ so that $l = r+\nu q+m$, and set $\mu = r+m$, we get:

PROPOSITION 6.9. *Let $\gamma \in \Gamma$ have rotation degree $\nu q > 0$ where q is the least positive element of $P_*(\Gamma)$. Then $\tilde{\chi}_1(X)(\gamma)$ is represented by the Hochschild cycle:*

$$\begin{aligned}&\sum_{n \geqslant 0} (-1)^n \operatorname{trace}(([\tilde{f}_n]t) \otimes \sum_{i=\mu}^{\mu+\nu q-1} ([\tilde{f}_n]t)^i) \\ &\qquad + \sum_{n \geqslant 0} (-1)^n \operatorname{trace}(([\tilde{f}_n]t) \otimes \sum_{i=0}^{\mu-1} ([\tilde{f}_n]t)^i)(1-\gamma^{-1})\end{aligned}$$

for any sufficiently large positive integer μ (dependent on γ). □

Remark 6.10. By Corollary 6.8, the same formula holds for γ of rotation degree 0; in that case, the first term in Proposition 6.9 is trivial.

If the subgroup $\Gamma' \subset \Gamma$ is finitely generated by $\gamma_1, \ldots, \gamma_r$ and if the number μ in Proposition 6.9 is taken to be the maximum of the numbers μ_i corresponding to γ_i, then we have an inner derivation $\mathscr{Y}: \Gamma' \to HH_1(\mathbf{Z}G)$ defined at the level of cycles by:

$$\mathscr{Y}(\gamma) = \sum_{n \geqslant 0} (-1)^n \operatorname{trace}(([\tilde{f}_n]t) \otimes \sum_{i=0}^{\mu-1} ([\tilde{f}_n]t)^i)(1-\gamma^{-1})\,.$$

This gives:

COROLLARY 6.11. *If* $i: \Gamma' \hookrightarrow \Gamma$ *is the inclusion of a finitely generated subgroup, there is an inner derivation* $\mathcal{Y}$ *such that for all* $\gamma \in \Gamma'$ *of rotation degree* $\nu q \geqslant 0$, $(\tilde{\mathrm{X}}_1(X) - \mathcal{Y})(\gamma)$ *is represented by the Hochschild cycle*

$$\sum_{n \geqslant 0} (-1)^n \operatorname{trace}(([\tilde{f}_n] t) \otimes \sum_{i=\mu}^{\mu+\nu q-1} ([\tilde{f}_n] t)^i)$$

which therefore depends only on the rotation degree of γ. *In particular, the derivation* $\tilde{\mathrm{X}}_1(X) - \mathcal{Y}$ *represents* $i^*(\tilde{\chi}_1(X))$. □

Now we can compute $\chi_1(X): \Gamma \to H_1(X) \equiv G_{\mathrm{ab}}$ using Definition A_1.

The automorphism $\theta: H \to H$ induces an automorphism $\theta_{\mathrm{ab}}: G_{\mathrm{ab}} \to G_{\mathrm{ab}}$. We identify G_{ab} with $\operatorname{coker}(\mathrm{id} - \theta_{\mathrm{ab}}) \times \mathbf{Z}$ by sending $ht^n \in G$ to $(\{h\}, -n)$. If $\gamma \in \Gamma$ has rotation degree 0, it follows from Corollary 6.8 that $\chi_1(X)(\gamma) = 0$. If $\gamma \in \Gamma$ has rotation degree $\nu q > 0$, we obtain $\chi_1(X)(\gamma)$ in two stages: first apply the augmentation, ε, to the right sides of the tensors in Proposition 6.9, yielding:

$$\sum_{n \geqslant 0} (-1)^n \operatorname{trace}(([\tilde{f}_n] t) \otimes \sum_{i=\mu}^{\mu+\nu q-1} [f_n^i]) \in C_1(\mathbf{Z}G, \mathbf{Z})$$

and then apply Proposition 2.1 to get:

$$\sum_{n \geqslant 0} \sum_{i=\mu}^{\mu+\nu q-1} (-1)^n A(\operatorname{trace}([\tilde{f}_n] t [f_n^i]))$$

$$= \sum_{n \geqslant 0} \sum_{i=\mu}^{\mu+\nu q-1} (-1)^n \left[A(\operatorname{trace}([\tilde{f}_n][f_n^i])) + \operatorname{trace}([f_n^i]) A(t) \right]$$

which simplifies to:

(6.12)

$$\chi_1(X)(\gamma) = \left(\sum_{n \geqslant 0} \sum_{i=\mu}^{\mu+\nu q-1} (-1)^n A(\operatorname{trace}([\tilde{f}_n][f_n^i])) \,,\; - \sum_{i=\mu}^{\mu+\nu q-1} L(f^i) \right)$$

$$\in \operatorname{coker}(\mathrm{id} - \theta_{\mathrm{ab}}) \times \mathbf{Z}\,.$$

Here, $L(f^i)$ is the Lefschetz number of f^i. Note that the matrix $A([\tilde{f}_n])$ has entries in $\operatorname{coker}(\mathrm{id} - \theta_{\mathrm{ab}})$, and for large μ the sequence $(L(f^\mu), \ldots, L(f^{\mu+\nu q-1}))$ is periodic since $f^r \simeq f^{r+q}$.

Summarizing:

THEOREM 6.13. *Let* $f: Z \to Z$ *be a cellular* π_1*-equivalence of a connected CW complex, and let* X *be the mapping torus* $T(Z, f)$.

(i) *if* f *is not an eventually coherent periodic homotopy idempotent, then* $\chi_1(X)(\gamma) = 0$ *for all* $\gamma \in \Gamma$;

(ii) *if* f *is an eventually coherent periodic homotopy idempotent of period* q, *and* $\gamma \in \Gamma$ *has rotation degree* $\nu q > 0$, *the two terms in (6.12) give the two factors of* $\chi_1(X)(\gamma) \in \operatorname{coker}(\mathrm{id} - \theta_{\mathrm{ab}}) \times \mathbf{Z}$; *if* γ *has rotation degree* 0, $\chi_1(X)(\gamma) = 0$. □

Remark. If f is not cellular then the above theorem can be applied to any cellular approximation of f. Since any two cellular approximations of f are homotopic, the corresponding mapping tori are homotopy equivalent. By homotopy invariance (Theorem 1.2), this procedure gives a well defined answer.

We get cleaner results when f is also a homotopy equivalence. If, in that case, q is the least positive element of $P_*(\Gamma)$, the proof of Theorem 6.3 shows that f satisfies the eventually coherent periodic homotopy idempotent property with $r = m = 0$; i.e. there is $N: \mathrm{id}_Z \simeq f^q$, and J making Diagram 6.1 commute with $m = 0$. The point here is that the inclusions $Z_n \to \bar{X}$ and $\cup_{n=0}^{q-1} M(f)_n \to \bar{X}$ are homotopy equivalences. Since it is now possible to "push" backwards as well as forwards in the telescope $\bar{X}$, we can also take $l = 0$ in the formula preceding Proposition 6.9. Thus we can take $\mu = 0$ in Proposition 6.9:

THEOREM 6.14. *If* f *is a homotopy equivalence and an eventually coherent periodic homotopy idempotent of period* q, *and* $\gamma \in \Gamma$ *has rotation degree* $\nu q > 0$, *then* $\tilde{\mathrm{X}}_1(X)(\gamma)$ *is represented by the Hochschild cycle*

$$\sum_{n \geqslant 0} (-1)^n \operatorname{trace}(([\tilde{f}_n]t) \otimes \sum_{i=0}^{\nu q - 1} ([\tilde{f}_n]t)^i) ;$$

and

$$\chi_1(X)(\gamma) = \left(\sum_{n \geqslant 0} \sum_{i=0}^{\nu q - 1} (-1)^n A(\operatorname{trace}([\tilde{f}_n][f_n^i])), \ - \sum_{i=0}^{\nu q - 1} L(f^i) \right) .$$

These formulas are determined by the rotation degree of γ. □

Example 6.15. Let $f = \mathrm{id}_Z$. Then $X = T(Z, \mathrm{id}_Z) = Z \times S^1$. Let $\nu \geqslant 0$. The ν-tumble, $\mathscr{R}_\nu$, represents an element of $\Gamma = \pi_1(\mathscr{E}(Z \times S^1), \mathrm{id})$ of rotation degree ν. By Theorem 6.14, we have:

$$\tilde{\mathrm{X}}_1(Z \times S^1)([\mathscr{R}_\nu]) = \chi(Z) T_1 \left(t \otimes \frac{1 - t^\nu}{1 - t} \right) .$$

This formula also holds for $\nu < 0$. It follows that $\chi_1(Z \times S^1)([\mathscr{R}_\nu]) = (0, -\chi(Z)\nu) = \chi(Z)\nu\{t\}$ where $\{t\} \in H_1(Z \times S^1) \cong H_1(Z) \oplus H_1(S^1)$ is the generator of the $H_1(S^1)$ summand determined by t.

There is a useful simplification of these formulas in the rational case. The identity

$$-\operatorname{trace}(([\tilde{f}_n]t)^{i+1}) \otimes 1 + (i+1)\operatorname{trace}([\tilde{f}_n]t \otimes ([\tilde{f}_n]t)^i)$$
$$= d\left(\sum_{j=1}^{i} \operatorname{trace}([\tilde{f}_n]t \otimes ([\tilde{f}_n]t)^{i-j+1} \otimes ([\tilde{f}_n]t)^{j-1}\right)$$

demonstrates that $\frac{1}{i+1}\operatorname{trace}(([\tilde{f}_n]t)^{i+1}) \otimes 1$ is homologous to $\operatorname{trace}([\tilde{f}_n]t \otimes ([\tilde{f}_n]t)^i)$. We can substitute in Proposition 6.9 and Theorem 6.14. Write $[\tilde{f}]$ for the matrix $\oplus_n(-1)^n[\tilde{f}_n]$. The matrix of the map $\tilde{f}^i$ is $\prod_{j=0}^{i-1}\theta^j([\tilde{f}])$, so $([\tilde{f}]t)^i = [\tilde{f}^i]t^i$. Thus we get:

THEOREM 6.16. $\tilde{X}_1(X; \mathbf{Q})(\gamma)$ *is represented by the Hochschild cycle*

$$\sum_{i=\mu+1}^{\mu+\nu q} \frac{1}{i}(\operatorname{trace}[\tilde{f}^i])t^i \otimes 1 + \left(\sum_{i=1}^{\mu} \frac{1}{i}(\operatorname{trace}[\tilde{f}^i])t^i \otimes 1\right)(1-\gamma^{-1})$$

for any sufficiently large positive integer μ *(dependent on* γ*). When* f *is also a homotopy equivalence then* $\tilde{X}_1(X; \mathbf{Q})(\gamma)$ *is represented by the Hochschild cycle*

$$\sum_{i=1}^{\nu q} \frac{1}{i}(\operatorname{trace}[\tilde{f}^i])t^i \otimes 1 . \qquad \square$$

Remark. The formula for $\tilde{X}_1(X; \mathbf{Q})(\gamma)$ above can be expressed in terms of the "reduced Reidemeister traces" of the iterates f^n, $n = 1, \ldots, \nu q$. This trace of f^n take values in the "reduced" 0-th Hochschild homology group of $\mathbf{Z}H$ with θ^n-twisted coefficients; see [GN$_2$, §5].

The computation of $\chi_1(X; \mathbf{Q})$ naturally leads one to consider the *homology Reidemeister trace* of a cellular map $f: Z \to Z$, denoted by $L^h(f)$. It is the element of $H_1(H) \cong H_{ab}$ given by

$$L^h(f) = \sum_{n \geqslant 0} (-1)^n A(\operatorname{trace}([\tilde{f}_n])) .$$

If k is a commutative ring of coefficients, let $L^h(f; k)$ denote the image of $L^h(f)$ under the homomorphism $H_1(H) \to H_1(H; k)$. Let $\bar{L}^h(f; k) \in \operatorname{coker}(\mathrm{id} - \theta_{ab} \otimes \mathrm{id}_k)$ denote the image of $L^h(f; k)$. It is easy to see that $L^h(f)$ and $\bar{L}^h(f; k)$ depend only on the homotopy class of f.

(Both $\bar{L}^h(f; k)$ and $L^h(f; k)$ have an interpretation in terms of Nielsen fixed point theory, but we will not make use of this.)

Theorem 6.16 together with the proof of (6.12) yields the following formula. For all sufficiently large μ:

$$\chi_1(X; \mathbf{Q})(\gamma) = \sum_{i=\mu+1}^{\mu+\nu q} \left(\tfrac{1}{i}\bar{L}^h(f^i; \mathbf{Q}), -L(f^i)\right).$$

Since this formula is valid for *all* sufficiently large μ, it is easy to see (because of periodicity and the appearance of the coefficients $\frac{1}{i}$) that:

Corollary 6.17. *For all sufficiently large* i, $\bar{L}^h(f^i; \mathbf{Q}) = 0$. □

Thus:

Corollary 6.18. *For all sufficiently large* μ:

$$\chi_1(X; \mathbf{Q})(\gamma) = \left(0, -\sum_{i=\mu+1}^{\mu+\nu q} L(f^i)\right).$$

In particular, if f *is also homotopy equivalence*

$$\chi_1(X; \mathbf{Q})(\gamma) = \left(0, -\sum_{i=0}^{\nu q-1} L(f^i)\right). \quad \square$$

7. More on groups of type $\mathscr{F}$

We consider in more detail the special case of the mapping torus of a homotopy equivalence of an aspherical complex.

Let H be an arbitrary group, let $\theta: H \to H$ be an automorphism, and let G be the semidirect product $\langle H, t \mid tht^{-1} = \theta(h) \text{ for all } h \in H\rangle$. Write $\mathrm{Fix}(\theta) = \{h \in H \mid \theta(h) = h\}$ and write $\langle x\rangle$ for the cyclic subgroup generated by $x \in G$. Let $\mathrm{Out}(H) = \mathrm{Aut}(H)/\mathrm{Inn}(H)$ be the group of outer automorphisms of H, i.e. the quotient of the group, $\mathrm{Aut}(H)$, of automorphisms of H by the normal subgroup $\mathrm{Inn}(H)$ of inner automorphisms.

Lemma 7.1. *If* θ *has infinite order in* $\mathrm{Out}(H)$, *then* $Z(G) = Z(H) \cap \mathrm{Fix}(\theta)$. *If* θ *has finite order* r *in* $\mathrm{Out}(H)$, *and* $h_0 \in H$ *is such that* $\theta^r(\cdot) = h_0(\cdot)h_0^{-1}$, *there are two cases:*

(1) *No positive power of* h_0 *lies in* $Z(H)\mathrm{Fix}(\theta)$. *Then* $Z(G) = Z(H) \cap \mathrm{Fix}(\theta)$.

(2) *Some positive power of* h_0 *lies in* $Z(H)\mathrm{Fix}(\theta)$. *Let* p *be the smallest positive integer such that* $h_0^{-p} \in Z(H)\mathrm{Fix}(\theta)$ *and let* $x = uh_0^{-p}t^{rp}$ *where* $u \in Z(H)$ *is such that* $uh_0^{-p} \in \mathrm{Fix}(\theta)$. *Then* $Z(G) = (Z(H) \cap \mathrm{Fix}(\theta))\langle x \rangle$.

Proof. Suppose $ht^m \in Z(G)$ where $h \in H$. Then $h\theta^m(h') = h'\theta^n(h)$ for every $h' \in H$, $n \in \mathbf{Z}$. In particular, taking $h' = 1$ and $n = 1$, $h \in \mathrm{Fix}(\theta)$. Taking h' arbitrary and $n = 1$, $\theta^m(h') = h^{-1}h'h$ for all $h' \in H$. Thus, if θ has infinite order in $\mathrm{Out}(H)$ and $ht^m \in Z(G)$ then $m = 0$ and $h \in Z(H)$. So $Z(G) \subset Z(H) \cap \mathrm{Fix}(\theta)$, and the reverse inclusion is clear.

If θ has finite order r in $\mathrm{Out}(H)$ and $ht^m \in Z(G)$, the above argument shows that $m = \nu r$ for some $\nu \in \mathbf{Z}$. So $\theta^{\nu r}(\cdot) = h^{-1}(\cdot)h = h_0^{\nu}(\cdot)h_0^{-\nu}$, implying $hh_0^{\nu} \in Z(H)$. Conversely, it is straightforward to show that any $ht^{\nu r}$ with $h \in \mathrm{Fix}(\theta) \cap h_0^{-\nu}Z(H)$ lies in $Z(G)$; hence: $Z(G) = \{ht^{\nu r} \in G \mid \nu \in \mathbf{Z}, h \in \mathrm{Fix}(\theta) \cap h_0^{-\nu}Z(H)\}$. If no positive power of h_0 lies in $Z(H)\mathrm{Fix}(\theta)$ then $ht^{\nu r} \in Z(G)$ if and only if $\nu = 0$ and $h \in Z(H) \cap \mathrm{Fix}(\theta)$. If some positive power of h_0 lies in $Z(H)\mathrm{Fix}(\theta)$, let p and u be as above. Then any $ht^{\nu r} \in Z(G)$ can be written as $(hh_0^{np}u^{-n})(uh_0^{-p}t^{rp})^n$ where $\nu = np$ (observe that $hh_0^{np}u^{-n} \in Z(H) \cap \mathrm{Fix}(\theta)$). $\square$

ADDENDUM 7.2. *If* θ *has finite order* r *in* $\mathrm{Out}(H)$ *and* $\theta^r(\cdot) = h_0(\cdot)h_0^{-1}$ *then*

$$Z(G) = \{ht^{\nu r} \in G \mid \nu \in \mathbf{Z},\ h \in \mathrm{Fix}(\theta),\ hh_0^{\nu} \in Z(H)\}.$$

Proof. In case (1) of Lemma 7.1 this is clear, and in case (2) it is part of the last proof. $\square$

We are very grateful to Peter Neumann for providing us with the proof of the following proposition which shows that (1) in Lemma 7.1 cannot occur.

PROPOSITION 7.3. *Let* $\theta: H \to H$ *be an automorphism whose image in* $\mathrm{Out}(H)$ *has finite order* r, *and let* $h_0 \in H$ *be such that* $\theta^r(\cdot) = h_0(\cdot)h_0^{-1}$. *Then* $h_0^r \in Z(H)\mathrm{Fix}(\theta)$.

Proof. Let $\theta^r(\cdot) = h_0(\cdot)h_0^{-1}$. Since $\theta^r\theta = \theta\theta^r$, we have $\theta(h_0) = h_0\zeta$ for some $\zeta \in Z(H)$. For $i = 0, \ldots, r-1$, let $\zeta_i = \theta^i(\zeta)$. The identity $h_0 = \theta^r(h_0)$ implies that $\zeta_0\zeta_1 \cdots \zeta_{r-1} = 1$. Define $x = h_0^r\zeta_0^{r-1}\zeta_1^{r-2} \cdots \zeta_{r-2}$. Then

$$\theta(x) = h_0^r\zeta_0^r\zeta_1^{r-1} \cdots \zeta_{r-2}^2\zeta_{r-1} = h_0^r\zeta_0^{r-1}\zeta_1^{r-2} \cdots \zeta_{r-2}$$

(the second equality uses $\zeta_{r-1} = (\zeta_0\zeta_1 \cdots \zeta_{r-2})^{-1}$ and the fact that the group generated by h_0 and $Z(H)$ is abelian). Thus $x \in \mathrm{Fix}(\theta)$ and so $h_0^r \in Z(H)\mathrm{Fix}(\theta)$. $\square$

Remark. In [GN$_3$] we called an automorphism θ as in Case (2) of Lemma 7.1 *special.* In view of Proposition 7.3, we abandon this terminology here.

Remark. The hypothesis of Proposition 7.3 yields a homomorphism $\mathbf{Z}/r\mathbf{Z} \to \mathrm{Out}(H)$ and hence a homomorphism $\mathbf{Z}/q\mathbf{Z} \to \mathrm{Out}(H)$ for any multiple q of r. There is a well-known obstruction $O_q \in H^3(\mathbf{Z}/q\mathbf{Z}, Z(H))$ whose vanishing is equivalent to the existence of an extension $1 \to H \to E \to \mathbf{Z}/q\mathbf{Z} \to 1$ with the given outer action. The content of Proposition 7.3 is that $O_{r^2} = 0$. For more on this, see [GN$_4$].

Combining Lemma 7.1 and Proposition 7.3, we have the following structure theorem for the center of the semidirect product G:

THEOREM 7.4. *If* θ *has infinite order in* $\mathrm{Out}(H)$, *then* $Z(G) = Z(H) \cap \mathrm{Fix}(\theta)$. *If* θ *has finite order* r *in* $\mathrm{Out}(H)$ *then* $Z(G) = (Z(H) \cap \mathrm{Fix}(\theta))\langle x\rangle$ *where* $x = uh_0^{-p}t^{rp}$, p *is the smallest positive integer dividing* r *such that* $h_0^{-p} \in Z(H)\mathrm{Fix}(\theta)$ *and* $u \in Z(H)$ *is such that* $uh_0^{-p} \in \mathrm{Fix}(\theta)$. $\square$

Definition 7.5. Let $\theta : H \to H$ be an automorphism whose image in $\mathrm{Out}(H)$ has finite order r, and let $h_0 \in H$ be such that $\theta^r(\cdot) = h_0(\cdot)h_0^{-1}$. The *period* of θ is the integer $q = pr$ where p is the least positive integer such $h_0^{-p} \in Z(H)\mathrm{Fix}(\theta)$.

Note that Proposition 7.3 guarantees that the period q exists. It is straightforward to show that q depends only on the image of θ in $\mathrm{Out}(H)$. From Definition 7.5, we have that r, the order of θ in $\mathrm{Out}(H)$, divides q and by Proposition 7.3 q divides r^2.

PROPOSITION 7.6. *Suppose* $\theta : H \to H$ *has finite order* m *in* $\mathrm{Aut}(H)$. *Then the period of* θ *divides* m.

Proof. Let $h_0 \in H$ be such that $\theta^r(\cdot) = h_0(\cdot)h_0^{-1}$ where r is the order of the image of θ in $\mathrm{Out}(H)$. Then $h_0^n \in Z(H) \subset Z(H)\mathrm{Fix}(\theta)$ where $n = m/r$. $\square$

We give some sufficient conditions for the period of an automorphism to coincide with its order in $\mathrm{Out}(H)$.

PROPOSITION 7.7. *Suppose* $\theta: H \to H$ *has finite order* r *in* $\mathrm{Out}(H)$, *the restriction of* θ *to* $Z(H)$ *is the identity, and* $Z(H)$ *and has no* l*-torsion for* l *dividing* r. *Then the period of* θ *is* r.

Proof. Let $\theta^r(\cdot) = h_0(\cdot)h_0^{-1}$. Using $\theta^r\theta = \theta\theta^r$, we have $\omega \equiv h_0\theta(h_0^{-1}) \in Z(H)$. The restriction of θ to $Z(H)$ is the identity so $\omega = \theta^j(\omega) = \theta^j(h_0)\theta^{j+1}(h_0^{-1})$ for any j. Thus $\omega^r = \prod_{j=0}^{r-1}\theta^j(h_0)\theta^{j+1}(h_0^{-1}) = h_0\theta^r(h_0^{-1}) = 1$. Since $Z(H)$ has no l-torsion for l dividing r, $\omega = 1$. Hence $h_0 \in \mathrm{Fix}(\theta)$. $\square$

A similar argument shows:

PROPOSITION 7.8. *Suppose* $\theta: H \to H$ *has finite odd order* r *in* $\mathrm{Out}(H)$ *and the restriction of* θ *to* $Z(H)$ *is given by* $h \mapsto h^{-1}$. *Then the period of* θ *is* r. $\square$

Let Z be a (not necessarily finite) $K(H, 1)$ complex and let $f: Z \to Z$ be a continuous map which induces θ (after choosing a basepoint and basepath). The homomorphism $(p_f)_*: G \to \mathbf{Z}$ of §6 is identified with $ht^m \mapsto -m$. Since $\Gamma = \pi_1(\mathscr{E}(Z), \mathrm{id}) \cong Z(G)$, the rotation degree homomorphism $P_*: Z(G) \to \mathbf{Z}$ is just the restriction of $(p_f)_*$. We immediately conclude from Theorem 7.4:

COROLLARY 7.9. *There is an an exact sequence* $0 \to Z(H) \cap \mathrm{Fix}(\theta) \to Z(G) \xrightarrow{P_*} \mathbf{Z}$ *such that* $P_*(Z(G)) = q\mathbf{Z}$ *where* $q = 0$ *if* θ *has infinite order in* $\mathrm{Out}(H)$ *and* $q > 0$ *is the period of* θ *if the image of* θ *has finite order in* $\mathrm{Out}(H)$. $\square$

Theorem 6.3 and the discussion preceding it yield:

PROPOSITION 7.10. *The map* f *is an eventually coherent periodic homotopy idempotent of period* $q > 0$ *if and only if* θ *has finite order in* $\mathrm{Out}(H)$ *and has period* $q > 0$. $\square$

Now suppose H is of type $\mathscr{F}$ so that we may take Z to be a finite $K(H, 1)$ complex. Assume f is cellular. Then $X = T(Z, f)$ is a finite $K(G, 1)$ complex. By (6.12) and Proposition 6.18,

THEOREM 7.11. *If* θ *has infinite order in* $\mathrm{Out}(H)$ *then* $\chi_1(G) = 0$. *If* θ *has finite order* r *in* $\mathrm{Out}(H)$ *and period* $q > 0$ *then*

$$\chi_1(G)(ht^{\nu q}) = \left(\sum_{n\geqslant 0}\sum_{i=0}^{\nu q-1}(-1)^n A(\mathrm{trace}([\tilde{f}_n][f_n^i])), -(q/r)\nu\sum_{i=0}^{r-1}L(f^i)\right)$$

and

$$\chi_1(G;\mathbf{Q})(ht^{\nu q}) = \left(0, -(q/r)\nu\sum_{i=0}^{r-1}L(f^i)\right) = (q/r)\nu\sum_{i=0}^{r-1}L(f^i)\{t\}$$

where $h \in \mathrm{Fix}(\theta) \cap h_0^{-\nu q/r}Z(H)$. □

Similarly, one can read off formulae for $\tilde{\mathrm{X}}_1(G)$ from Theorem 6.14 and the rational version from Theorem 6.16.

8. Outer automorphisms of groups of type $\mathscr{F}$

In this section we apply the preceding theory to prove the following theorem which relates the algebraic topology of an automorphism $\theta: H \to H$ of a group H of type $\mathscr{F}$ such that θ has finite order in $\mathrm{Out}(H)$ to the fixed group of θ.

Theorem 8.1. *Let H be a group of type $\mathscr{F}$ which has the Weak Bass Property over* $\mathbf{Q}$. *Suppose that $\theta: H \to H$ is an automorphism whose order in* $\mathrm{Out}(H)$ *is* $r \geqslant 1$. *If the sum of the Lefschetz numbers* $\sum_{i=0}^{r-1}L(\theta^i)$ *is non-zero then* $Z(H) \cap \mathrm{Fix}(\theta) = (1)$.

Before proving this we note that the quantity $\sum_{i=0}^{r-1}L(\theta^i)$ appearing above has the following interpretation:

Proposition 8.2. $\sum_{i=0}^{r-1}L(\theta^i)$ *is r times the Euler characteristic of the θ-invariant part of the homology of H, i.e.,*

$$\sum_{i=0}^{r-1}L(\theta^i) = r\sum_{j\geqslant 0}(-1)^j \mathrm{rank}\,\ker(\mathrm{id}-\theta_j: H_j(H)\to H_j(H)) .$$

Proof. By elementary linear algebra, for any square complex matrix A with $A^r = I$ we have $\mathrm{trace}(\sum_{i=0}^{r-1}A^i) = r\dim\ker(I-A)$. The conclusion easily follows. □

Proof of Theorem 8.1. Let G be the semidirect product $G = H \times_\theta T$ where T is infinite cyclic. By Lemma 8.7, below, G also has the WBP over $\mathbf{Q}$. Applying Theorem 7.11 to G, we have that $\chi_1(G;\mathbf{Q}) \neq 0$. By

Theorem 5.4, $Z(G)$ is infinite cyclic. By Corollary 7.9 there is an exact sequence $1 \to Z(H) \cap \mathrm{Fix}(\theta) \to Z(G) \xrightarrow{P_*} q\mathbf{Z} \to 1$ where the period of θ, q, is positive. It follows that $Z(H) \cap \mathrm{Fix}(\theta) = (1)$. $\square$

If $\chi(H) \neq 0$ then $Z(H) = (1)$ by Proposition 2.4 and consequently $Z(H) \cap \mathrm{Fix}(\theta) = (1)$ in this case. If $\chi(H) = L(\theta^0) = 0$ then $\sum_{i=0}^{r-1} L(\theta^i) = \sum_{i=1}^{r-1} L(\theta^i)$. These observations yield the following corollaries of Theorem 8.1:

COROLLARY 8.3. *Let H be a group of type $\mathscr{F}$ which has the WBP over* $\mathbf{Q}$. *Suppose that $\theta: H \to H$ is an automorphism of order* 2 *in* $\mathrm{Out}(H)$. *If $L(\theta) \neq 0$ then* $Z(H) \cap \mathrm{Fix}(\theta) = (1)$. $\square$

COROLLARY 8.4. *Let H be a group of type $\mathscr{F}$ which has the WBP over* $\mathbf{Q}$. *Suppose $Z(H) \neq (1)$, the automorphism $\theta: H \to H$ has finite order r in* $\mathrm{Out}(H)$ *and the restriction of θ to $Z(H)$ is the identity. Then* $\sum_{i=1}^{r-1} L(\theta^i) = 0$.

Proof. Since the restriction of θ to $Z(H)$ is the identity, $Z(H) \cap \mathrm{Fix}(\theta) = Z(H) \neq (1)$. $\square$

An automorphism which has finite order in $\mathrm{Out}(H)$ may have infinite order in $\mathrm{Aut}(H)$. If θ has finite order in $\mathrm{Aut}(H)$, the Weak Bass Property hypothesis can be dispensed with in Theorem 8.1 and Corollary 8.3:

PROPOSITION 8.5. *Let H be a group of type $\mathscr{F}$. Suppose that $\theta: H \to H$ has finite order in* $\mathrm{Aut}(H)$ *and $L(\theta) \neq 0$. Then* $Z(H) \cap \mathrm{Fix}(\theta) = (1)$.

Proof. Let $\omega \in Z(H) \cap \mathrm{Fix}(\theta)$. We use the terminology of [Br]. Let Z be a finite $K(H, 1)$. Choose an essential fixed point, v, of $f: Z \to Z$ (inducing θ) as the basepoint of Z. There is a homotopy $K: f \simeq f$ such that $K(v, \cdot)$ represents ω. The fixed point v is K-related to some fixed point u of f [Br, p. 92]. Hence, for some $s > 0$, v is J-related to v, where J is the s-fold concatenation $K \star \cdots \star K$. Then there exists $\sigma \in H$ such that $\omega^s = \sigma\theta(\sigma^{-1})$; compare [G]. As in the proof of Proposition 7.7, we get $\omega^{rs} = \prod_{i=0}^{r-1} \theta^i(\sigma\theta(\sigma^{-1})) = 1$, so $\omega = 1$. $\square$

Note that $\sum_{i=1}^{r-1} L(\theta^i) \neq 0$ implies one of the $L(\theta^i)$'s is non-zero. Since $\mathrm{Fix}(\theta) \subset \mathrm{Fix}(\theta^i)$ for $i \geqslant 0$, we recover Theorem 8.1 (but without the Bass Conjecture hypothesis) in the special case where θ has finite order in $\mathrm{Aut}(H)$.

The remainder of this section is devoted to the proof of Lemma 8.7 used above.

LEMMA 8.6. *Suppose that the group H has the WBP over $\mathbf{Q}$. Let T be an infinite cyclic group. Then the product group $H \times T$ also has the WBP over $\mathbf{Q}$.*

Proof. Let $G = H \times T$. Identify H with $H \times \{1\} \subset G$. We use the notation of § 5. By Schafer's theorem [Sch, p. 224] applied to the normal subgroup $H \subset G$, the image of $T_0: K_0(\mathbf{Q}G) \to HH_0(\mathbf{Q}G)$ lies in $HH_0(\mathbf{Q}G)_H$. Let $p: G \to H$ be the projection homomorphism. There is a commutative diagram:

$$\begin{array}{ccccc} K_0(\mathbf{Q}G) & \xrightarrow{T_0} & HH_0(\mathbf{Q}G)_H & \xrightarrow{\varepsilon_*} & \mathbf{Q} \\ {\scriptstyle p_*}\downarrow & & {\scriptstyle p_*}\downarrow & & \| \\ K_0(\mathbf{Q}H) & \xrightarrow{T_0} & HH_0(\mathbf{Q}H) & \xrightarrow{\varepsilon_*} & \mathbf{Q} \end{array}$$

Write $HH_0(\mathbf{Q}G)_H = HH_0(\mathbf{Q}G)_{C(1)} \oplus HH_0(\mathbf{Q}G)''_H$ where $HH_0(\mathbf{Q}G)''_H$ is the direct sum of the $HH_0(\mathbf{Q}G)_{C(g)}$'s over $C(g) \in c(H) - \{C(1)\}$; also, $HH_0(\mathbf{Q}H) = HH_0(\mathbf{Q}H)_{C(1)} \oplus HH_0(\mathbf{Q}H)'$. By hypothesis, H has the WBP over $\mathbf{Q}$, i.e. the composite

$$K_0(\mathbf{Q}H) \xrightarrow{T_0} HH_0(\mathbf{Q}H) \to HH_0(\mathbf{Q}H)' \xrightarrow{\varepsilon_*} \mathbf{Q}$$

is zero. Since $p_*(HH_0(\mathbf{Q}G)_{C(1)}) \subset HH_0(\mathbf{Q}H)_{C(1)}$ and $p_*(HH_0(\mathbf{Q}G)''_H) \subset HH_0(\mathbf{Q}H)'$, the conclusion follows. □

LEMMA 8.7. *Suppose that the group H has the WBP over $\mathbf{Q}$ and that $\theta: H \to H$ is an automorphism whose image in the group of outer automorphisms of H has finite order. Then the semidirect product $H \times_\theta T$ also has the WBP over $\mathbf{Q}$.*

Proof. Let $G = H \times_\theta T \equiv \langle H, t \mid tht^{-1} = \theta(h) \text{ for } h \in H \rangle$. Let n be the order of θ in the group outer automorphisms of H. Then the subgroup G' of G generated by H and t^n is isomorphic to $H \times T$; furthermore, G' is normal and of finite index, n, in G. There is a "transfer" homomorphism trans: $HH_0(\mathbf{Q}G) \to HH_0(\mathbf{Q}G')$ defined as follows. Given $g \in G$, we can write $gt^i = t^{\sigma(i)} g_i$ for $i = 0, \ldots, n-1$ where $g_i \in G'$ and σ is a permutation of $\{0, \ldots, n-1\}$. Let $\mathrm{Fix}(\sigma) = \{i \mid \sigma(i) = i\}$. Then $\mathrm{trans}(C(g)) = \sum_{i \in \mathrm{Fix}(\sigma)} C(g_i)$. Observe that if $g \in G'$ then $\mathrm{Fix}(\sigma) = \{0, \ldots, n-1\}$

because G' is normal in G. In particular, $\varepsilon_*(\mathrm{trans}(C(g))) = n$ if $g \in G'$. There is a commutative diagram:

$$\begin{array}{ccc} K_0(\mathbf{Q}G) & \xrightarrow{T_0} & HH_0(\mathbf{Q}G) \\ \mathrm{res} \downarrow & & \mathrm{trans} \downarrow \\ K_0(\mathbf{Q}G') & \xrightarrow{T_0} & HH_0(\mathbf{Q}G') \end{array}$$

where $\mathrm{res}: K_0(\mathbf{Q}G) \to K_0(\mathbf{Q}G')$ is obtained by regarding a projective $\mathbf{Q}G$ module as a projective $\mathbf{Q}G'$ module; see [Bass] for details concerning the finite index transfer.

Recall that $HH_0(\mathbf{Q}G) = HH_0(\mathbf{Q}G)_H \oplus HH_0(\mathbf{Q}G)'_H$ where $HH_0(\mathbf{Q}G)'_H$ is the direct sum of the summands $HH_0(\mathbf{Q}G)_{C(g)}$ corresponding to the conjugacy classes not represented by elements of H. By Schafer's theorem [Sch, p. 224] applied to the normal subgroup $H \subset G$, the image of $T_0: K_0(\mathbf{Q}G) \to HH_0(\mathbf{Q}G)$ lies in $HH_0(\mathbf{Q}G)_H$. Thus we can replace $HH_0(\mathbf{Q}G)$ with $HH_0(\mathbf{Q}G)_H$ in the above diagram and obtain the commutative diagram:

$$\begin{array}{ccccc} K_0(\mathbf{Q}G) & \xrightarrow{T_0} & HH_0(\mathbf{Q}G)_H & \xrightarrow{\varepsilon_*} & \mathbf{Q} \\ \mathrm{res} \downarrow & & \mathrm{trans} \downarrow & & \times n \downarrow \\ K_0(\mathbf{Q}G') & \xrightarrow{T_0} & HH_0(\mathbf{Q}G') & \xrightarrow{\varepsilon_*} & \mathbf{Q} \end{array}$$

(the right square commutes because $H \subset G'$ and because of the observation made above). Write $HH_0(\mathbf{Q}G)_H = HH_0(\mathbf{Q}G)_{C(1)} \oplus HH_0(\mathbf{Q}G)''_H$ where $HH_0(\mathbf{Q}G)''_H$ is the direct sum of the $HH_0(\mathbf{Q}G)_{C(g)}$'s over $C(g) \in c(H) - \{C(1)\}$; also, $HH_0(\mathbf{Q}G') = HH_0(\mathbf{Q}G')_{C(1)} \oplus HH_0(\mathbf{Q}G')'$. Then $\mathrm{trans}(HH_0(\mathbf{Q}G)_{C(1)}) \subset HH_0(\mathbf{Q}G')_{C(1)}$ and $\mathrm{trans}(HH_0(\mathbf{Q}G)''_H) \subset HH_0(\mathbf{Q}G')'$. By Lemma 8.6, G' has the WBP over $\mathbf{Q}$, i.e. the composite $K_0(\mathbf{Q}G') \xrightarrow{T_0} HH_0(\mathbf{Q}G') \to HH_0(\mathbf{Q}G')' \xrightarrow{\varepsilon_*} \mathbf{Q}$ is zero. The conclusion follows from the above diagram. $\square$

9. Trace formulae for homological intersections

The goal of this section is to prove a "trace formula" (Theorem 9.13) for the homological intersection of the graph of a map $F: M \times Y \to M$ with the graph of the projection map $p: M \times Y \to M$ where Y is a closed oriented manifold and M is a compact oriented manifold. This result will be applied in § 10 to complete the proof of Theorem 1.1.

In what follows, all homology and cohomology groups will have coefficients in a field $\mathbf{F}$. Recall that we use Dold's sign conventions [D2] for cup, cap and cross products.

Let M be a compact n-dimensional manifold with boundary ∂M. Assume M is oriented over $\mathbf{F}$ with fundamental class $[M] \in H_n(M, \partial M)$. Let $V \subset M$ be an open collar of ∂M and $M' = M - V$. Let $\Delta \subset M \times M$ be the diagonal and let

$$(M \times M, M' \times \partial M) \overset{j}{\hookrightarrow} (M \times M, M \times M - \Delta) \quad \text{and}$$
$$(M \times M, M' \times \partial M) \overset{i}{\hookrightarrow} (M \times M, M \times \partial M)$$

be the inclusions. Since i is a homotopy equivalence of pairs it induces an isomorphism $i^*: H^*(M \times M, M \times \partial M) \to H^*(M \times M, M' \times \partial M)$. We define the *diagonal cohomology class* $D_M \in H^n(M \times M, M \times \partial M)$ by $D_M \equiv (i^*)^{-1} j^*(T_M)$ where

$$T_M \in H^n(M \times M, M \times M - \Delta)$$

is the Thom class of M (see [Sp, §6.2] where T_M is called an *orientation* for M).

There is a slant product $H^i(M \times M, M \times \partial M) \otimes H_j(M, \partial M) \xrightarrow{/} H^{i-j}(M)$, see [MS, p. 125]. The reader should be aware that the sign conventions for cup, cap and cross products used in [MS] coincide with those of [D2] but differ from those of [Sp]. A straightforward adaptation of the proof of [MS, Lemma 11.9], where the case $\partial M = \emptyset$ is treated, shows that the fundamental class of M and the diagonal cohomology class of M are related by:

PROPOSITION 9.1. $D_M/[M] = 1 \in H^0(M)$. □

For each $k \geqslant 0$, choose a basis $\{b_j^k \mid j = 1, \ldots, N(k)\}$ for $H_k(M)$. Let $\{\bar{b}_j^k \mid j = 1, \ldots, N(k)\}$, be the corresponding dual basis for $H^k(M)$, i.e. $\langle \bar{b}_i^k, b_j^k \rangle = \delta_{ij}$ (Kronecker delta). For $k \geqslant 0$, define $d_j^{n-k} \in H^{n-k}(M, \partial M)$, $j = 1, \ldots, N(k)$, by $b_j^k = d_j^{n-k} \cap [M]$. The proof of [MS, Theorem 11.11] carries over directly to show:

PROPOSITION 9.2. $D_M = \sum_{k \geqslant 0} (-1)^k \sum_{i=1}^{N(k)} \bar{b}_i^k \times d_i^{n-k}$. □

Let Y be a parameter space (Y is not required to be a manifold). Let $F: M \times Y \to M$ be a map. For $\alpha \in H_q(Y)$, define $f_{ij}^k(\alpha) \in \mathbf{F}$ by $F_*(b_j^k \times \alpha) = \sum_{i=1}^{N(k+q)} f_{ij}^k(\alpha) b_i^{k+q}$.

The Künneth Theorem allows us to write

$$F^*(\bar{b}_j^k) = \sum_{s=0}^{k} \sum_{l=1}^{N(s)} \bar{b}_l^s \times \omega(k,j,s,l)$$

where $\omega(k,j,s,l) \in H^{k-s}(Y)$.

LEMMA 9.3. $f_{ij}^k(\alpha) = (-1)^{qk}\langle \omega(k+q,i,k,j), \alpha \rangle$.

Proof. We have:

$$\begin{aligned} f_{ij}^k(\alpha) &= \langle \bar{b}_i^{k+q}, F_*(b_j^k \times \alpha)\rangle = \langle F^*(\bar{b}_i^{k+q}), b_j^k \times \alpha \rangle \\ &= \sum_{s=0}^{k+q} \sum_{l=1}^{N(s)} \langle \bar{b}_l^s \times \omega(k+q,i,s,l), b_j^k \times \alpha \rangle \\ &= \sum_{s=0}^{k+q} \sum_{l=1}^{N(s)} (-1)^{(k+q-s)k} \langle \bar{b}_l^s \cap b_j^k, \omega(k+q,i,s,l) \cap \alpha \rangle \\ &= (-1)^{qk}\langle \omega(k+q,i,k,j), \alpha \rangle . \qquad \square \end{aligned}$$

Let $\bar{F}: M \times Y \to M \times M$ be defined by $\bar{F}(m,y) = (F(m,y), m)$ and let $p: M \times Y \to M$ be projection. We define the *intersection invariant of F* to be the degree 0 homomorphism $\bar{I}(F): H_*(Y) \to H_*(M)$ given by $\bar{I}(F)(\alpha) = p_*(\bar{F}^*(D_M) \cap ([M] \times \alpha)) \in H_q(M)$ where $\alpha \in H_q(Y)$.

PROPOSITION 9.4. *For any* $\alpha \in H_q(Y)$,

$$\bar{I}(F)(\alpha) = \sum_{k \geqslant 0} (-1)^k \sum_{j=1}^{N(k)} \bar{b}_j^k \cap F_*(b_j^k \times \alpha) .$$

Proof. We have:

$$\begin{aligned} \bar{F}^*(\bar{b}_j^k \times d_j^{n-k}) &= F^*(\bar{b}_j^k) \cup (d_j^{n-k} \times 1) \\ &= \left(\sum_{s=0}^{k} \sum_{l=1}^{N(s)} \bar{b}_l^s \times \omega(k,j,s,l) \right) \cup (d_j^{n-k} \times 1) \\ &= \sum_{s=0}^{k} \sum_{l=1}^{N(s)} (-1)^{(k-s)(n-k)} (\bar{b}_l^s \cup d_j^{n-k}) \times \omega(k,j,s,l) . \end{aligned}$$

Now $(\bar{b}_l^s \cup d_j^{n-k}) \cap [M] = \bar{b}_l^s \cap (d_j^{n-k} \cap [M]) = \bar{b}_l^s \cap b_j^k$ and thus

$$\bar{F}^*(\bar{b}_j^k \times d_j^{n-k}) \cap ([M] \times \alpha)$$

$$= \sum_{s=0}^{k} \sum_{l=1}^{N(s)} (-1)^{(k-s)(n-k)} (-1)^{(k-s)n} ((\bar{b}_l^s \cup d_j^{n-k}) \cap [M]) \times (\omega(k,j,s,l) \cap \alpha)$$

$$= \sum_{s=0}^{k} \sum_{l=1}^{N(s)} (-1)^{(k-s)k} (\bar{b}_l^s \cap b_j^k) \times (\omega(k,j,s,l) \cap \alpha) .$$

Using Proposition 9.2 and the above identity, we obtain:

$$\bar{F}^*(D_M) \cap ([M] \times \alpha)$$

$$= \sum_{k \geqslant 0} (-1)^k \sum_{j=1}^{N(k)} \sum_{s=0}^{k} \sum_{l=1}^{N(s)} (-1)^{(k-s)k} (\bar{b}_l^s \cap b_j^k) \times (\omega(k,j,s,l) \cap \alpha) .$$

Now $p_*((\omega(k,j,s,l) \cap \alpha) \times (\bar{b}_l^s \cap b_j^k)) = 0$ unless $k - s = q$. Thus

$$p_*(\bar{F}^*(D_M) \cap ([M] \times \alpha))$$

$$= \sum_{k \geqslant 0} (-1)^k \sum_{j=1}^{N(k)} \sum_{l=1}^{N(k-q)} (-1)^{qk} \langle \omega(k,j,k-q,l), \alpha \rangle (\bar{b}_l^{k-q} \cap b_j^k) .$$

Since $\bar{b}_l^{k-q} = 0$ for $k - q < 0$, we can rewrite the above expression using the index variable $r = k - q$ as:

$$p_*(\bar{F}^*(D_M) \cap ([M] \times \alpha))$$

(9.5)

$$= \sum_{r \geqslant 0} (-1)^r \sum_{j=1}^{N(r+q)} \sum_{l=1}^{N(r)} (-1)^{qr} \langle \omega(r+q,j,r,l), \alpha \rangle (\bar{b}_l^r \cap b_j^{r+q}) .$$

Using Lemma 9.3,

$$\sum_{k \geqslant 0} (-1)^k \sum_{j=1}^{N(k)} \bar{b}_j^k \cap F_*(b_j^k \times \alpha) = \sum_{k \geqslant 0} (-1)^k \sum_{j=1}^{N(k)} \sum_{i=1}^{N(k+q)} f_{ij}^k(\alpha)(\bar{b}_j^k \cap b_i^{k+q})$$

$$= \sum_{k \geqslant 0} (-1)^k \sum_{j=1}^{N(k)} \sum_{i=1}^{N(k+q)} (-1)^{qk} \langle \omega(k+q,i,k,j), \alpha \rangle (\bar{b}_j^k \cap b_i^{k+q}) .$$

Clearly, this last expression is the same as (9.5). □

We define the *diagonal homology class* $\Delta_M \in H_n(M \times M, \partial M \times M)$ by $\Delta_M \equiv \Delta_*([M])$ where Δ is the diagonal map $\Delta(x) = (x, x)$ regarded as a map of pairs $\Delta: (M, \partial M) \to (M \times M, \partial M \times M)$.

The homology class Δ_M can be expressed in terms of a basis for homology and Poincaré duality. Let $\{b_j^k\}$, $\{\bar{b}_j^k\}$ and $\{d_j^{n-k}\}$ be as in the discussion preceding Proposition 9.4. Let $a_j^{n-k} \equiv \bar{b}_j^k \cap [M] \in H_{n-k}(M, \partial M)$, $j = 1, \ldots, N(k)$.

PROPOSITION 9.6. $\Delta_M = \sum_{k \geqslant 0} \sum_{i=1}^{N(k)} (-1)^{k(n-k)} a_i^{n-k} \times b_i^k$. □

Proof. Without loss of generality, we can assume that M is connected. Observe

$$\begin{aligned} d_i^{n-k} \cap a_j^{n-k} &= d_i^{n-k} \cap (\bar{b}_j^k \cap [M]) = (d_i^{n-k} \cup \bar{b}_j^k) \cap [M] \\ &= (-1)^{(n-k)k} (\bar{b}_j^k \cup d_i^{n-k}) \cap [M] \\ &= (-1)^{(n-k)k} \bar{b}_j^k \cap (d_i^{n-k} \cap [M]) = (-1)^{(n-k)k} \bar{b}_j^k \cap b_i^k \\ &= (-1)^{(n-k)k} \delta_{ij} b_1^0 \end{aligned}$$

where δ_{ij} is Kronecker's delta.

By the Künneth formula, we can write

$$\Delta_M = \sum_{k \geqslant 0} \sum_{i=1}^{N(k)} \sum_{j=1}^{N(k)} c_{ij}^k a_i^{n-k} \times b_j^k$$

where $c_{ij}^k \in \mathbf{F}$. We have

$$\begin{aligned} (d_r^{n-l} \times \bar{b}_s^l) \cap \Delta_M &= (d_r^{n-l} \times \bar{b}_s^l) \cap \Delta_*([M]) = \Delta_*(\Delta^*(d_r^{n-l} \times \bar{b}_s^l) \cap [M]) \\ &= \Delta_*((d_r^{n-l} \cup \bar{b}_s^l) \cap [M]) \\ &= (-1)^{(n-l)l} \Delta_*((\bar{b}_s^l \cup d_r^{n-l}) \cap [M]) \\ &= (-1)^{(n-l)l} \Delta_*(\bar{b}_s^l \cap (d_r^{n-l} \cap [M])) \\ &= (-1)^{(n-l)l} \Delta_*(\bar{b}_s^l \cap b_r^l) = (-1)^{(n-l)l} \delta_{rs} b_1^0 \times b_1^0 . \end{aligned}$$

Now, $(d_r^{n-l} \times \bar{b}_s^l) \cap (a_i^{n-k} \times b_j^k) = 0$ whenever $l \neq k$ and

$$\begin{aligned} (d_r^{n-l} \times \bar{b}_s^l) \cap (a_i^{n-l} \times b_j^l) &= (-1)^{l(n-l)} (d_r^{n-l} \cap a_i^{n-l}) \times (\bar{b}_s^l \cap b_j^l) \\ &= \delta_{ri} \delta_{sj} b_1^0 \times b_1^0 . \end{aligned}$$

It follows that $c_{rs}^l = (-1)^{l(n-l)} \delta_{rs}$. $\square$

Up to sign, the diagonal homology and the diagonal cohomology classes are Poincaré dual:

PROPOSITION 9.7. $D_M \cap ([M] \times [M]) = (-1)^n \Delta_M$.

Proof. Observe that

$$\begin{aligned} (b_i^k \times d_i^{n-k}) \cap ([M] \times [M]) &= (-1)^{(n-k)k} (b_i^k \cap [M]) \times (d_i^{n-k} \cap [M]) \\ &= (-1)^{(n-k)k} a_i^{n-k} \times b_i^k . \end{aligned}$$

Using the formula for D_M given by Proposition 9.2,

$$
\begin{aligned}
D_M \cap ([M] \times [M]) &= \sum_{k \geqslant 0} (-1)^k \sum_{i=1}^{N(k)} (\bar{b}_i^k \times d_i^{n-k}) \cap ([M] \times [M]) \\
&= \sum_{k \geqslant 0} (-1)^k \sum_{i=1}^{N(k)} (-1)^{(n-k)k} a_i^{n-k} \times b_i^k \\
&= (-1)^n \sum_{k \geqslant 0} \sum_{i=1}^{N(k)} (-1)^{k(n-k)} a_i^{n-k} \times b_i^k \\
&= (-1)^n \Delta_M \quad \text{by Proposition 9.6.} \quad \square
\end{aligned}
$$

Until now Y has been an arbitrary parameter space. For what follows we assume that Y is a closed q-dimensional manifold which is oriented over **F**. Let $[Y] \in H_n(Y)$ be the fundamental class. Define $\mathrm{Gr}(p): M \times Y \to M \times Y \times M$ and $\mathrm{Gr}(F): M \times Y \to M \times Y \times M$ by $\mathrm{Gr}(p)(m, y) = (m, y, m)$ and $\mathrm{Gr}(F)(m, y) = (m, y, F(m, y))$. Define homology classes

$$
\begin{aligned}
A &= \mathrm{Gr}(p)_*([M] \times [Y]) \in H_{n+q}(M \times Y \times M, M \times Y \times \partial M)\,, \\
B &= \mathrm{Gr}(F)_*([M] \times [Y]) \in H_{n+q}(M \times Y \times M, \partial M \times Y \times M)\,.
\end{aligned}
$$

We define the *intersection product* $A \bullet B \in H_q(M \times Y \times M)$ as follows. Let

$$
\begin{aligned}
\delta_1 &: H^n(M \times Y \times M, \partial M \times Y \times M) \to H_{n+q}(M \times Y \times M, M \times Y \times \partial M) \\
\delta_2 &: H^n(M \times Y \times M, M \times Y \times \partial M) \to H_{n+q}(M \times Y \times M, \partial M \times Y \times M)
\end{aligned}
$$

be the Poincaré duality isomorphisms for the manifold triad $(M \times Y \times M; M \times Y \times \partial M, \partial M \times Y \times M)$ given by cap product with $[M] \times [Y] \times [M]$. Then

$$
A \bullet B \equiv (\delta_1^{-1}(B) \cup \delta_2^{-1}(A)) \cap [M] \times [Y] \times [M]\,.
$$

Definition 9.8. The *graph intersection invariant of* F is $\theta'(F) \equiv (p_1)_*(A \bullet B) \in H_q(M)$ where $p_1: M \times Y \times M \to M$ is projection to the first M factor.

Remark 9.9. The graph intersection invariant of F can be obtained geometrically using transversality. Suppose F has no fixed points on $\partial M \times Y$. Then the boundaries of the embedded submanifolds $\mathrm{Gr}(p)(M \times Y) \subset M \times Y \times M$ and $\mathrm{Gr}(F)(M \times Y) \subset M \times Y \times M$ are disjoint and so these submanifolds may be made transverse via an ambient isotopy of the identity which leaves a neighborhood of the boundary of $M \times Y \times M$ (pointwise) fixed. The set theoretic intersection of the perturbed submanifolds is a closed orientable manifold of dimension q which we orient using the "intersection orientation" taken in the order: the perturbed $\mathrm{Gr}(p)(Y \times M)$ first followed by the perturbed $\mathrm{Gr}(F)(Y \times M)$. By Proposition 11.13 of [D_2, §VIII], the resulting oriented manifold is a cycle representing $A \bullet B$. Projecting this cycle to M via p_1 yields a representative of $\theta'(F)$.

The isomorphisms δ_1^{-1} and δ_2^{-1} can be described explicitly using the slant product. Let $(Z; \partial_1 Z, \partial_2 Z)$ be a compact oriented manifold triad and $K = \partial\partial_1 Z = \partial\partial_2 Z$. Since

$$(Z - \partial_1 Z, \partial_2 Z - K) \times (Z - \partial_2 Z, \partial_1 Z - K) \subset (Z \times Z, Z \times Z - \Delta),$$
$$(Z - \partial_2 Z, \partial_1 Z - K) \times (Z - \partial_1 Z, \partial_2 Z - K) \subset (Z \times Z, Z \times Z - \Delta)$$

there are slant product pairings:

$$H^i(Z \times Z, Z \times Z - \Delta) \otimes H_j(Z - \partial_2 Z, \partial_1 Z - K) \xrightarrow{/} H^{i-j}(Z - \partial_1 Z, \partial_2 Z - K)$$
$$H^i(Z \times Z, Z \times Z - \Delta) \otimes H_j(Z - \partial_1 Z, \partial_2 Z - K) \xrightarrow{/} H^{i-j}(Z - \partial_2 Z, \partial_1 Z - K)$$

By the existence of collars, the inclusions $(Z - \partial_2 Z, \partial_1 Z - K) \hookrightarrow (Z, \partial_1 Z)$ and $(Z - \partial_1 Z, \partial_2 Z - K) \hookrightarrow (Z, \partial_2 Z)$ are homotopy equivalences and so we obtain pairings:

$$H^i(Z \times Z, Z \times Z - \Delta) \otimes H_j(Z, \partial_1 Z) \xrightarrow{/} H^{i-j}(Z, \partial_2 Z),$$
$$H^i(Z \times Z, Z \times Z - \Delta) \otimes H_j(Z, \partial_2 Z) \xrightarrow{/} H^{i-j}(Z, \partial_1 Z).$$

Let $m = \dim Z$. The inverse to the Poincaré duality isomorphisms

$$\delta_1 : H^{m-j}(Z, \partial_2 Z) \to H_j(Z, \partial_1 Z), \quad \delta_1(x) = x \cap ([Z] \times [Z])$$
$$\delta_2 : H^{m-j}(Z, \partial_1 Z) \to H_j(Z, \partial_2 Z), \quad \delta_2(x) = x \cap ([Z] \times [Z])$$

are explicitly given by $\delta_1^{-1}(y) = (-1)^{m(m-j)} T_Z / y$ and $\delta_2^{-1}(y) = (-1)^{m(m-j)} T_Z / y$ where $T_Z \in H^m(Z \times Z, Z \times Z - \Delta)$ is the Thom class of Z (see [MS, p. 135]).

PROPOSITION 9.10. $\theta'(F) = \bar{I}(F)([Y])$.

Proof. Without loss of generality, we may assume Y is connected. Let $S : M \times M \to M \times M$ be the "interchange map", i.e. $S(x, y) = (y, x)$. Now $S_*([M] \times [M]) = (-1)^n [M] \times [M]$ and so by Proposition 9.7, $\Delta_M = D_M \cap S_*([M] \times [M]) = S_*(S^*(D_M) \cap ([M] \times [M]))$. Hence $S_*(\Delta_M) = S^*(D_M) \cap ([M] \times [M])$. Using the inverse to the Poincaré duality isomorphism, we have $T_{M \times M} / S_*(\Delta_M) = S^*(D_M)$.

Define $\hat{F} = S \circ \bar{F}$. Then $\bar{F} = S \circ \hat{F}$. Also note that $p = p_1' \circ \hat{F}$ where $p_1' : M \times M \to M$ is projection to the first factor. From the definition of $\bar{I}(F)$,

$$
\begin{aligned}
\bar{I}(F)([Y]) &= p_*(\bar{F}^*(D_M) \cap ([M] \times [Y])) \\
&= (p_1')_* \hat{F}_* (\hat{F}^* S^*(D_M) \cap ([M] \times [Y])) \\
&= (p_1')_* (S^*(D_M) \cap \hat{F}_*([M] \times [Y])) \qquad (9.11) \\
&= (p_1')_* (T_{M \times M}/S_*(\Delta_M) \cap \hat{F}_*([M] \times [Y])) \\
&= (p_1'')_* (T_{M \times M} \cap \hat{F}_*([M] \times [Y]) \times S_*(\Delta_M))
\end{aligned}
$$

where p_1'' is projection to the first "M" factor.

Let $I': M \times Y \times M \to M \times M \times Y$ and

$$I'': M \times M \times Y \times M \times M \times Y \to M \times M \times M \times M \times Y \times Y$$

be the "interchange maps" given by $I'(m_1, y, m_2) = (m_1, m_2, y)$ and $I(m_1, m_2, y, m_3, m_4, y') = (m_1, m_2, m_3, m_4, y, y')$. Let $I = I'' \circ (I' \times I')$. Then $I^*(T_{M \times M} \times T_Y) = T_{M \times Y \times M}$ and

$$
\begin{aligned}
\theta'(F) &= (p_1)_*((\delta_1^{-1}(B) \cup \delta_2^{-1}(A)) \cap ([M] \times [Y] \times [M])) \\
&= (p_1)_*(\delta_1^{-1}(B) \cap (\delta_2^{-1}(A) \cap ([M] \times [Y] \times [M]))) \\
&= (p_1)_*(\delta_1^{-1}(B) \cap A) \\
&= (-1)^{qn}(p_1)_*((T_{M \times Y \times M}/B) \cap A) \qquad (9.12) \\
&= (-1)^{qn}(p_1^3)_*(T_{M \times Y \times M} \cap (A \times B)) \\
&= (-1)^{qn}(p_1^4)_* I_*(I^*(T_{M \times M} \times T_Y) \cap (A \times B)) \\
&= (-1)^{qn}(p_1^4)_*(T_{M \times M} \times T_Y \cap I_*(A \times B))
\end{aligned}
$$

where p_1^4 and $p_1^3 = p_1^4 \circ I$ are projections to the first "M" factor. We have

$$
\begin{aligned}
I'_*(A) &= \mathrm{Gr}(p)_*([Y] \times [M]) = S_*(\Delta_M) \times [Y] \\
I'_*(B) &= \mathrm{Gr}(F)_*([Y] \times [M]) = \hat{F}_*([Y] \times [M]) \times [y_0] + \beta
\end{aligned}
$$

where $[y_0] \in H_0(Y)$ is represented by $y_0 \in Y$ and β is a finite sum of the form $\beta = \sum_i v_i \times u_i$ with $u_i \in H_{n_i}(Y)$, $n_i \geqslant 1$, and $v_i \in H_{n+q-n_i}(M \times M, \partial M \times M)$. It follows that:

$$
\begin{aligned}
I_*(A \times B) = {} & (-1)^{q(n+q)} S_*(\Delta_M) \times \hat{F}_*([Y] \times [M]) \times [Y] \times [y_0] \\
& + \sum_i (-1)^{q(n+q-n_i)} S_*(\Delta_M) \times v_i \times [Y] \times u_i .
\end{aligned}
$$

Since $T_Y \cap ([Y] \times u_i)$ lies in homology of degree $n_i > 0$,

$$
\begin{aligned}
& (p_1^4)_*((T_{M \times M} \times T_Y) \cap (S_*(\Delta_M) \times v_i \times [Y] \times u_i)) \\
&= (-1)^{q(q-n_i)}(p_1^4)_*((T_{M \times M} \cap (S_*(\Delta_M) \times v_i)) \times (T_Y \cap ([Y] \times u_i))) = 0 .
\end{aligned}
$$

Using (9.12),

$$\begin{aligned}
\theta'(F) &= (-1)^{qn}(p_1^4)_*(T_{M\times M}\times T_Y \cap I_*(A\times B))\\
&= (-1)^{qn}(-1)^{q(n+q)}(p_1^4)_*((T_{M\times M}\times T_Y)\cap(S_*(\Delta_M)\times \hat{F}_*([Y]\\
&\quad \times [M])\times [Y]\times [y_0]))\\
&= (-1)^{q}(-1)^{q}(p_1^4)_*((T_{M\times M}\cap(S_*(\Delta_M)\times \hat{F}_*([Y]\times[M])))\\
&\quad \times (T_Y\cap([Y]\times[y_0])))\\
&= (p_1^4)_*((T_{M\times M}\cap(S_*(\Delta_M)\times \hat{F}_*([Y]\times[M])))\times([y_0]\times[y_0]))\\
&= (p_1'')_*(T_{M\times M}\cap(S_*(\Delta_M)\times \hat{F}_*([Y]\times[M])))\\
&= \bar{I}(F)([Y]) \quad \text{by (9.11).} \quad \square
\end{aligned}$$

Combining Propositions 9.4 and 9.10 yields:

THEOREM 9.13 (Trace Formula). *The graph intersection invariant is given by:*

$$\theta'(F) = \sum_{k\geq 0}(-1)^k \sum_{j=1}^{N(k)} \bar{b}_j^k \cap F_*(b_j^k \times [Y]). \quad \square$$

Remark. It is easy to check that Theorem 9.13 remains valid over a principal ideal domain R in place of the coefficient field $\mathbf{F}$, provided we assume that $H_*(M; R)$ is a free R-module.

10. PROOFS OF THEOREMS 1.1 AND 1.5

In this section we prove Theorems 1.1 and 1.5 which assert the equivalence, under appropriate hypotheses, of the four definitions of the first order Euler characteristic introduced in §1.

Proof of Theorem 1.1 (ii). Let M be a compact connected oriented PL or smooth n-manifold with boundary (as well as being the underlying simplicial complex of a compatible triangulation). Using Definition A_1, we are to show that $\chi_1(M)(\gamma) = -\theta(\gamma)$; the case of other coefficient rings R will then follow immediately. Fattening if necessary, assume $n \geq 4$.

Let $J: M\times I \to M$ be a homotopy from id_M to a map j, such that the graph of $J|_{M\times[\frac{1}{2},1]}$ meets the graph of $p|_{M\times[\frac{1}{2},1]}$ transversely in $|\chi(M)|$ arcs; this can be achieved by classical techniques of cancelling unnecessary pairs of fixed points. Note that j will then have precisely $|\chi(M)|$ fixed points, all transverse and having the same fixed point index.

Denote by $\bar{F}^\gamma$, the concatenated homotopy $J^{-1} \star F^\gamma \star J$. It is clear that, using Definition A_1, $\mathrm{trace}(\tilde{\partial}_{k+1} D_k^\gamma) = \mathrm{trace}(\tilde{\partial}_{k+1} \bar{D}_k^\gamma)$, since the new contributions cancel one another. By perturbing rel $M \times \{0, 1\}$, we may assume that the graph of $\bar{F}^\gamma$ meets the graph of p transversely, and that $\mathrm{Fix}(\bar{F}^\gamma)$ consists of circles in $\mathring{M} \times (0, 1)$ and $|\chi(M)|$ arcs in $\mathring{M} \times I$ joining $\mathring{M} \times \{0\}$ to $\mathring{M} \times \{1\}$. It may be assumed (see [GN_1, § 6(B)]) that $\bar{F}^\gamma$ is cellular with respect to suitable triangulations of M and $M \times I$.

If $\chi(M) = 0$, there are no arcs. In that case, the required geometric arguments are to be found in [GN_1, § 6]; and Definitions A_1 and C_1 are indeed equivalent. (The point is that in [GN_1] there is a precise sense in which contributions to the fixed point set associated with $M \times \{0, 1\}$ are ignored, so that when such points are present, i.e. when $\chi(M) \neq 0$, something more must be said, and will now be said.)

Suppose $\chi(M) \neq 0$. $\bar{F}^\gamma$ is a homotopy from j to j. By our constructions, since j is homotopic to id_M and has the least possible number of transverse fixed points, all those fixed points are in the same fixed point class, (in the sense of classical Nielsen fixed point theory [Br], [J]). Moreover, the arcs are all in the same fixed point class of $\bar{F}^\gamma$ in the analogous sense defined in [GN_1]. By symmetry, if an arc meets $(x, 0)$ then an arc meets $(x, 1)$, but perhaps a different arc. However, since all the arcs are in the same fixed point class, the methods of [Di] allow us to perturb $\bar{F}^\gamma$ rel $M \times \{0, 1\}$ so that, for the perturbed map, an arc meeting $(x, 0)$ also meets $(x, 1)$. The arc $\beta(t) \equiv \bar{F}^\gamma(x, t)$ is homotopically trivial, for if the arc of fixed points α joins $(x, 0)$ to $(x, 1)$ then β is homotopic to $(\bar{F}^\gamma \circ \alpha)(p \circ \alpha)^{-1}$. Thus the methods of [Di] allow us to perturb $\bar{F}^\gamma$ further so that α is replaced by a circle of fixed points missing $M \times \{0, 1\}$ together with an arc of fixed points coinciding with β. Thus these arcs contribute zero to $\theta_R(\gamma)$. So, again, the argument in [GN_1, § 6] shows, that Definitions A_1 and C_1 are equivalent: the trace formula in Definition A_1 describes the homology class of the circles.

Summarizing, we have proved Part (ii) of Theorem 1.1. □

We prove Part (i) of Theorem 1.1 by first showing that Definitions B_1 and C_1 agree when X is a compact oriented manifold. Then, using the already proved Part (ii), we establish the equivalence of Definitions A_1 and B_1.

The trace formula in Definition B_1 was introduced by Knill in [Kn]. As we remarked in § 1, it is independent of basis. Moreover, it is a straightforward exercise to show that it is a homotopy invariant.

Proof of Theorem 1.1(i). Let X be a finite CW complex, as in §1. By homotopy invariance of the formulas in A_1 and B_1, we may assume the attaching maps in X are polyhedral. Therefore we may PL embed X in some $\mathbf{R}^n$ as a strong deformation retract of a compact codimension 0 PL submanifold, M, e.g. a regular neighborhood. Now, any F^γ as in §1 can be extended to map $M \times S^1 \to X \hookrightarrow M$ by precomposing with $r \times \mathrm{id}$ where $r: M \to X$ is a strong deformation retraction. By Remark 9.9 and Theorem 9.13, Definitions B_1 and C_1 are equivalent for M. By Theorem 1.1(ii), Definitions A_1 and C_1 are equivalent for M. Hence, using homotopy invariance, Definitions A_1 and B_1 are equivalent for X. □

In [Kn] there also appears an "intersection class", whose definition we now recall. (Actually, the context in [Kn] is much more general: we only extract what we need.)

Throughout the remainder of this section, all homology and cohomology groups will have coefficients in the principal ideal domain R. Let M be a compatibly oriented, compact, codimension 0, PL submanifold $i: M \hookrightarrow \mathbf{R}^n$. Let $F: M \times S^1 \to M$ be such that $\mathrm{Fix}(F) \cap \partial M \times S^1 = \emptyset$. Let $[M \times S^1] \in H_{n+1}(M \times S^1, \partial M \times S^1)$ be the fundamental class of $M \times S^1$ and let $[\mathbf{R}^n]$ be the generator of $H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\})$ determined by the orientation. Following Leray [Le] and Dold [D_1], Knill defines the intersection class of F to be the image, $I_R(F)$, of $[M \times S^1]$ under the following composition:

$$H_{n+1}((M, \partial M) \times S^1) \to H_{n+1}(M \times S^1, M \times S^1 - \mathrm{Fix}(F)) \xrightarrow{(i \circ p - i \circ F, F)_*}$$
$$H_{n+1}((\mathbf{R}^n, \mathbf{R}^n - \{0\}) \times M) \xrightarrow{\cong} H_1(M)$$

where $p: M \times S^1 \to M$ is projection and $H_{n+1}((\mathbf{R}^n, \mathbf{R}^n - \{0\}) \times M) \xrightarrow{\cong} H_1(M)$ is the inverse of the isomorphism $H_1(M) \xrightarrow{\cong} H_{n+1}((\mathbf{R}^n, \mathbf{R}^n - \{0\}) \times M)$, $y \mapsto [\mathbf{R}^n] \times y$.

We make use of the following special case of [Kn, Theorem 1]:

THEOREM 10.1. *Suppose $H_*(M)$ is a free R-module. Then*

$$-I_R(F) = \sum_{k \geqslant 0} (-1)^{k+1} \sum_j \bar{b}_j^k \cap F_*(b_j^k \times [S^1])$$

where $[S^1] \in H_1(S^1)$ is the fundamental class and where for each $k \geqslant 0$, $\{b_j^k\}$ is a basis for $H_k(X)$ with corresponding dual basis, $\{\bar{b}_j^k\}$, for $H^k(X)$. The cap product is taken with Dold's sign convention. □

Proof of Theorem 1.5. We show $-p_*\tau(\bar{\Phi}^\gamma)_*([S^1])$ coincides with Definition B_1. As in the proof of Theorem 1.1(i) above, we may assume that X is a compact polyhedron which is PL embedded in some $\mathbf{R}^n$ as a strong deformation retract of a compact codimension 0 PL submanifold, M. Extend Φ^γ to a map $\Psi^\gamma : M \times S^1 \to X \hookrightarrow M$ by precomposing with $r \times \mathrm{id}$ where $r: M \to X$ is a strong deformation retraction. The homotopy invariance of Definition B_1 and Theorem 10.1 imply that $-I_R(\Psi^\gamma) = \chi_1(X, R)(\gamma)$. By [$D_3$, (3.3)] and [BG, §9], $I_R(\Psi^\gamma)$ coincides with $p_*\tau(\bar{\Phi}^\gamma)_*([S^1])$. □

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