3. Constructions of Chaotic Group Actions

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suppose that groups $G_i, i \in I$ act faithfully with all orbits finite on the Hausdorff spaces $M_i, i \in I$ respectively. Now to each space M_i, add an additional isolated element x_i and denote $\tilde{M}_i$ the union $M_i \cup \{x_i\}$. Then we define an action of G_i on $\tilde{M}_i$ by using the action of G_i on M_i and making x_i a fixed point. Clearly G_i acts faithfully with all orbits finite on $\tilde{M}_i$. Now let M denote the subset of the infinite product $\prod_{i \in I} \tilde{M}_i$ composed of all elements $(y_i)_{i \in I}$ for which only finitely many of the y_i are different from x_i. We equip M with the topology induced by the product topology. Then clearly the infinite direct product $\prod_{i \in I} G_i$ acts faithfully with all orbits finite on M.

Finally Part (f) is similar to Part (e); suppose that a group G acts chaotically on a space M and that H is a finite group. Then there is a natural action of the wreath product $G \mathrm{Wr} H$ on the space $M \times H$, where H is given the discrete topology (see [H]). It is easy to see that this action is chaotic. □

The following groups are known to be residually finite: Fuchsian groups [LS], the mapping class groups of compact Riemann surfaces [G], arithmetic groups [Se] and the group of p-adic integers [We]. It would be interesting to find natural chaotic actions of these groups.

3. Constructions of Chaotic Group Actions

First recall that there are many examples of chaotic $\mathbf{Z}$-actions; that is, chaotic homeomorphisms. Perhaps the most basic example is that of the Anosov diffeomorphisms of tori and infranilmanifolds (see [Sm], [Mann]); these maps are chaotic since their periodic points are dense [BR] and by Anosov's closing lemma (see for instance [Sh]), they are transitive on their nonwandering set. (The Anosov diffeomorphisms of tori are just the linear hyperbolic maps; that is, linear maps with no eigenvalues on the unit circle.) Similarly, the pseudo-Anosov maps of compact surfaces are also chaotic (see Exposé 9 in [FLP] and the diagrams in [Mañ], pages 111-116).

Let us now give some general results.

THEOREM 2. *Consider a Hausdorff space* M *and the group* $\mathrm{Hom}(M)$ *of homeomorphisms of* M. *Then one has:*

(a) If there are group inclusions

$$G \leqslant H \leqslant K \leqslant \mathrm{Hom}(M)$$

then the action of H *on* M *is chaotic if the actions of* G *and* K *on* M *are chaotic.*

(b) *If* $G \leqslant H \leqslant \mathrm{Hom}(M)$ *and* G *has finite index in* H *and if the action of* G *on* M *is chaotic, then the action of* H *on* M *is chaotic.*

(c) *If* M *is locally compact and if* $\mathrm{Hom}(M)$ *is given the compact-open topology, then the action of* G *on* M *is chaotic if and only if the action on* M *of the closure* $\bar{G}$ *of* G *in* $\mathrm{Hom}(M)$ *is chaotic.*

Proof. In Part (a), notice that if a point $x \in M$ has finite orbit under K, then x obviously has finite orbit under H. So if the action of K has finite orbits dense, then the action of H has finite orbits dense. On the other hand, if the action of G is topologically transitive, then clearly the action of H is also topologically transitive. So Part (a) holds. Part (b) is similar to Part (a).

In Part (c), again if the action of $\bar{G}$ has finite orbits dense, then the action of G has finite orbits dense. Now suppose that the action of $\bar{G}$ is topologically transitive. Let U and V be two non-empty open subsets of M. Then there exists $g \in \bar{G}$ such that $g(U) \cap V$ is non-empty. Let x be an element of $U \cap g^{-1}(V)$ and let Θ be the open subset of $\bar{G}$ composed of elements that send x into V. Then $g \in \Theta$ and since G is dense in $\bar{G}$, there exists $h \in G \cap \Theta$. So $h(U) \cap V$ is non-empty and hence the action of G is topologically transitive.

Conversely, if M is locally compact, then the natural map $\mathrm{Hom}(M) \times M \to M$ is continuous. So, if a point $x \in M$ has finite orbit under G, then since G is dense in $\bar{G}$, one has that $G(x)$ is dense in $\bar{G}(x)$. Hence $\bar{G}(x)$ is finite. So if the action of G has finite orbits dense, then the action of $\bar{G}$ has finite orbits dense. Finally, if the action of G is topologically transitive, then obviously so too is the action of $\bar{G}$. □

4. Manifolds That Admit Chaotic Group Actions

Chaotic homeomorphisms of the 2-dimensional disc can be constructed as follows. Starting with any Anosov diffeomorphism of the torus $\mathbf{T}^2$, one can quotient by the map $\sigma : x \mapsto -x$, to obtain a chaotic homeomorphism on the sphere $\mathbf{S}^2$. (This map was used in [Wa], p. 140 to show that expansiveness is not preserved under semi-conjugation.) Then, by blowing up the origin to a circle, one obtains a chaotic homeomorphism on the closed disc. Unfortunately this latter homeomorphism is not the identity on the boundary. This can be rectified by making a slight modification of the above construction. Instead of starting with an Anosov diffeomorphism of $\mathbf{T}^2$, one starts with linked twist map [D1] of the torus $\mathbf{T}^2$. A linked twist map is an