

# 3. Constructions of Chaotic Group Actions

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suppose that groups  $G_i, i \in I$  act faithfully with all orbits finite on the Hausdorff spaces  $M_i, i \in I$  respectively. Now to each space  $M_i$ , add an additional isolated element  $x_i$  and denote  $\tilde{M}_i$  the union  $M_i \cup \{x_i\}$ . Then we define an action of  $G_i$  on  $\tilde{M}_i$  by using the action of  $G_i$  on  $M_i$  and making  $x_i$  a fixed point. Clearly  $G_i$  acts faithfully with all orbits finite on  $\tilde{M}_i$ . Now let  $M$  denote the subset of the infinite product  $\prod_{i \in I} \tilde{M}_i$  composed of all elements  $(y_i)_{i \in I}$  for which only finitely many of the  $y_i$  are different from  $x_i$ . We equip  $M$  with the topology induced by the product topology. Then clearly the infinite direct product  $\prod_{i \in I} G_i$  acts faithfully with all orbits finite on  $M$ .

Finally Part (f) is similar to Part (e); suppose that a group  $G$  acts chaotically on a space  $M$  and that  $H$  is a finite group. Then there is a natural action of the wreath product  $G \text{Wr} H$  on the space  $M \times H$ , where  $H$  is given the discrete topology (see [H]). It is easy to see that this action is chaotic.  $\square$

The following groups are known to be residually finite: Fuchsian groups [LS], the mapping class groups of compact Riemann surfaces [G], arithmetic groups [Se] and the group of  $p$ -adic integers [We]. It would be interesting to find natural chaotic actions of these groups.

### 3. CONSTRUCTIONS OF CHAOTIC GROUP ACTIONS

First recall that there are many examples of chaotic  $\mathbf{Z}$ -actions; that is, chaotic homeomorphisms. Perhaps the most basic example is that of the Anosov diffeomorphisms of tori and infranilmanifolds (see [Sm], [Mann]); these maps are chaotic since their periodic points are dense [BR] and by Anosov's closing lemma (see for instance [Sh]), they are transitive on their nonwandering set. (The Anosov diffeomorphisms of tori are just the linear hyperbolic maps; that is, linear maps with no eigenvalues on the unit circle.) Similarly, the pseudo-Anosov maps of compact surfaces are also chaotic (see Exposé 9 in [FLP] and the diagrams in [Mañ], pages 111-116).

Let us now give some general results.

**THEOREM 2.** *Consider a Hausdorff space  $M$  and the group  $\text{Hom}(M)$  of homeomorphisms of  $M$ . Then one has:*

(a) *If there are group inclusions*

$$G \leq H \leq K \leq \text{Hom}(M)$$

*then the action of  $H$  on  $M$  is chaotic if the actions of  $G$  and  $K$  on  $M$  are chaotic.*

- (b) If  $G \leq H \leq \text{Hom}(M)$  and  $G$  has finite index in  $H$  and if the action of  $G$  on  $M$  is chaotic, then the action of  $H$  on  $M$  is chaotic.
- (c) If  $M$  is locally compact and if  $\text{Hom}(M)$  is given the compact-open topology, then the action of  $G$  on  $M$  is chaotic if and only if the action on  $M$  of the closure  $\bar{G}$  of  $G$  in  $\text{Hom}(M)$  is chaotic.

*Proof.* In Part (a), notice that if a point  $x \in M$  has finite orbit under  $K$ , then  $x$  obviously has finite orbit under  $H$ . So if the action of  $K$  has finite orbits dense, then the action of  $H$  has finite orbits dense. On the other hand, if the action of  $G$  is topologically transitive, then clearly the action of  $H$  is also topologically transitive. So Part (a) holds. Part (b) is similar to Part (a).

In Part (c), again if the action of  $\bar{G}$  has finite orbits dense, then the action of  $G$  has finite orbits dense. Now suppose that the action of  $\bar{G}$  is topologically transitive. Let  $U$  and  $V$  be two non-empty open subsets of  $M$ . Then there exists  $g \in \bar{G}$  such that  $g(U) \cap V$  is non-empty. Let  $x$  be an element of  $U \cap g^{-1}(V)$  and let  $\Theta$  be the open subset of  $\bar{G}$  composed of elements that send  $x$  into  $V$ . Then  $g \in \Theta$  and since  $G$  is dense in  $\bar{G}$ , there exists  $h \in G \cap \Theta$ . So  $h(U) \cap V$  is non-empty and hence the action of  $G$  is topologically transitive.

Conversely, if  $M$  is locally compact, then the natural map  $\text{Hom}(M) \times M \rightarrow M$  is continuous. So, if a point  $x \in M$  has finite orbit under  $G$ , then since  $G$  is dense in  $\bar{G}$ , one has that  $G(x)$  is dense in  $\bar{G}(x)$ . Hence  $\bar{G}(x)$  is finite. So if the action of  $G$  has finite orbits dense, then the action of  $\bar{G}$  has finite orbits dense. Finally, if the action of  $G$  is topologically transitive, then obviously so too is the action of  $\bar{G}$ .  $\square$

#### 4. MANIFOLDS THAT ADMIT CHAOTIC GROUP ACTIONS

Chaotic homeomorphisms of the 2-dimensional disc can be constructed as follows. Starting with any Anosov diffeomorphism of the torus  $\mathbf{T}^2$ , one can quotient by the map  $\sigma: x \mapsto -x$ , to obtain a chaotic homeomorphism on the sphere  $\mathbf{S}^2$ . (This map was used in [Wa], p. 140 to show that expansiveness is not preserved under semi-conjugation.) Then, by blowing up the origin to a circle, one obtains a chaotic homeomorphism on the closed disc. Unfortunately this latter homeomorphism is not the identity on the boundary. This can be rectified by making a slight modification of the above construction. Instead of starting with an Anosov diffeomorphism of  $\mathbf{T}^2$ , one starts with linked twist map [D1] of the torus  $\mathbf{T}^2$ . A linked twist map is an