

# Proof of Jacobi Sum Congruence Via Stickelberger

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which he explicitly points out is valid in all characteristics. Thus a proof of Stickelberger's congruence for all finite fields via the Gross-Koblitz formula is justified.)

### PROOF OF JACOBI SUM CONGRUENCE VIA STICKELBERGER

We now want to show that not only does Theorem 1 follow from Theorem 2, but Theorem 2 follows from Theorem 1, so the two theorems are equivalent. Some preliminary results will be required before the (tedious) proof is presented.

For  $n \in \mathbf{N}$ , write

$$n = c_0 + c_1 p + \cdots + c_d p^d, \quad 0 \leq c_i \leq p - 1.$$

From [3, Chapter IX],

$$\text{ord}_p(n!) = \frac{n - (c_0 + \cdots + c_d)}{p - 1}, \quad \frac{n!}{(-p)^{\text{ord}_p(n!)}} \equiv c_0! \cdot \cdots \cdot c_d! \pmod{p}.$$

Note neither equation requires  $c_d \neq 0$ . We define

$$S_p(n) \stackrel{\text{def}}{=} c_0 + \cdots + c_d, \quad H_p(n) \stackrel{\text{def}}{=} c_0! \cdot \cdots \cdot c_d!,$$

and note neither of these definitions requires  $c_d \neq 0$ . One sees easily that for any  $n \in \mathbf{N}$ ,  $n \equiv S_p(n) \pmod{p - 1}$ , and for  $n_1, \dots, n_t \in \mathbf{N}$ ,

$$\text{ord}_p \left( \frac{(n_1 + \cdots + n_t)!}{n_1! \cdots n_t!} \right) = \frac{S_p(n_1) + \cdots + S_p(n_t) - S_p(n_1 + \cdots + n_t)}{p - 1}.$$

For  $x \in \mathbf{R}$ , let  $\langle x \rangle$  denote the fractional part of  $x$ . For  $b \in \mathbf{Z}$ , let  $b \equiv b' \pmod{q - 1}$  where  $0 \leq b' < q - 1$ , so that  $\langle \frac{b}{q - 1} \rangle = \frac{b'}{q - 1}$ . Define

$$s_q(b) = S_p(b'), \quad h_q(b) = H_p(b'),$$

so  $s_q$  and  $h_q$  are just the extensions of  $S_p$  and  $H_p$  from  $\{b : 0 \leq b < q - 1\}$  by  $(q - 1)$ -periodicity. From [7, p. 10],

$$s_q(b) = (p - 1) \sum_{0 \leq i \leq f - 1} \left\langle \frac{p^i b}{q - 1} \right\rangle.$$

Since  $\text{ord}_{\mathfrak{P}}(\zeta_p - 1) = 1$ , Stickelberger's congruence can be written for all  $a$  in  $\mathbf{Z}$  as

$$\frac{G(\omega_p^{-a})}{(\zeta_p - 1)^{s_q(a)}} \equiv \frac{1}{h_q(a)} \pmod{\mathfrak{P}}.$$

LEMMA 2. For  $r, m \in \mathbf{Z}^+$ , and  $b_1, \dots, b_r \in \mathbf{Z}$ ,

$$\left\langle \frac{b_1}{m} \right\rangle + \dots + \left\langle \frac{b_r}{m} \right\rangle \geq \left\langle \frac{b_1 + \dots + b_r}{m} \right\rangle.$$

If  $b_1 + \dots + b_r \equiv 0 \pmod m$  and some  $b_j \not\equiv 0 \pmod m$  then

$$\left\langle \frac{b_1}{m} \right\rangle + \dots + \left\langle \frac{b_r}{m} \right\rangle \geq 1.$$

*Proof.* Let  $b_j \equiv b'_j \pmod m$ , where  $0 \leq b'_j < m$ . Then  $b'_1 + \dots + b'_r \geq 0$ , so since  $x \geq \langle x \rangle$  for  $x \geq 0$ ,

$$\begin{aligned} \left\langle \frac{b_1}{m} \right\rangle + \dots + \left\langle \frac{b_r}{m} \right\rangle &= \frac{b'_1 + \dots + b'_r}{m} \geq \left\langle \frac{b'_1 + \dots + b'_r}{m} \right\rangle \\ &= \left\langle \frac{b_1 + \dots + b_r}{m} \right\rangle. \end{aligned}$$

If  $b_1 + \dots + b_r \equiv 0 \pmod m$  then  $(b'_1 + \dots + b'_r)/m \in \mathbf{N}$ . If some  $b_j \not\equiv 0 \pmod m$  then  $b'_j > 0$ , so  $(b'_1 + \dots + b'_r)/m \in \mathbf{Z}^+$ , hence is  $\geq 1$ .  $\square$

COROLLARY 1. Let  $0 \leq k_1, \dots, k_r < q - 1$  with  $k_1 + \dots + k_r \geq q - 1$ , so  $r \geq 2$  and at least two  $k_j > 0$ . Then

$$s_q(k_1) + \dots + s_q(k_r) \begin{cases} > s_q(k_1 + \dots + k_r) & \text{if } k_1 + \dots + k_r \not\equiv 0 \pmod{q-1} \\ > f(p-1) & \text{if } k_1 + \dots + k_r \equiv 0 \pmod{q-1}, \\ & > q-1 \\ \geq f(p-1) & \text{if } k_1 + \dots + k_r = q-1. \end{cases}$$

*Proof.* From above,

$$s_q(k_1) + \dots + s_q(k_r) = (p-1) \sum_{0 \leq i \leq f-1} \left( \left\langle \frac{p^i k_1}{q-1} \right\rangle + \dots + \left\langle \frac{p^i k_r}{q-1} \right\rangle \right).$$

If  $k_1 + \dots + k_r \not\equiv 0 \pmod{q-1}$ , applying Lemma 2 to  $p^i k_1, \dots, p^i k_r$  shows that each addend is  $\geq \left\langle \frac{p^i(k_1 + \dots + k_r)}{q-1} \right\rangle$ , with strict inequality when  $i = 0$  by hypothesis, since

$$\left\langle \frac{k_1}{q-1} \right\rangle + \dots + \left\langle \frac{k_r}{q-1} \right\rangle = \frac{k_1 + \dots + k_r}{q-1} > 1 \geq \left\langle \frac{k_1 + \dots + k_r}{q-1} \right\rangle.$$

If  $k_1 + \dots + k_r \equiv 0 \pmod{q-1}$  then by Lemma 2 each addend is  $\geq 1$ , with strict inequality when  $i = 0$  if  $k_1 + \dots + k_r > q - 1$ .  $\square$

We now state a more general version of Lemma 1, with a different notation that will be better suited for what follows.

LEMMA 3. For  $k_1, \dots, k_r \in \mathbf{Z}$  with some  $k_j \not\equiv 0 \pmod{q-1}$ ,

$$J(\omega_p^{-k_1}, \dots, \omega_p^{-k_r}) = \begin{cases} \frac{G(\omega_p^{-k_1}) \cdots G(\omega_p^{-k_r})}{G(\omega_p^{-(k_1 + \cdots + k_r)})} & \text{if } k_1 + \cdots + k_r \not\equiv 0 \pmod{q-1} \\ \frac{1}{q} G(\omega_p^{-k_1}) \cdots G(\omega_p^{-k_r}) & \text{if } k_1 + \cdots + k_r \equiv 0 \pmod{q-1}. \end{cases}$$

*Proof.* Use [6, Chapter 8, Theorem 3] and its corollaries, keeping in mind the differences mentioned between that book and this paper on various definitions.  $\square$

*Proof that Theorem 1 implies Theorem 2.* We have  $0 \leq k_1, \dots, k_r < q-1$  with some  $k_j > 0$ , so if the second case of Lemma 3 holds, then  $r \geq 2$  and at least two  $k_j$  are  $> 0$ . From the multinomial coefficient manipulations at the end of the proof of Theorem 2, if  $k_1 + \cdots + k_r > q-1$  then

$$\frac{(k_1 + \cdots + k_r)!}{k_1! \cdots k_r!} \equiv 0 \pmod{p}. \quad (*)$$

Thus to prove Theorem 1 implies Theorem 2 we are led to the following four cases:

Case 1:  $k_1 + \cdots + k_r > q-1$ ,  $k_1 + \cdots + k_r \not\equiv 0 \pmod{q-1}$

Case 2:  $k_1 + \cdots + k_r > q-1$ ,  $k_1 + \cdots + k_r \equiv 0 \pmod{q-1}$

Case 3:  $k_1 + \cdots + k_r = q-1$

Case 4:  $0 < k_1 + \cdots + k_r < q-1$ .

We will prove Theorem 2 from Theorem 1 by establishing the congruence of Theorem 2 modulo  $\mathfrak{P}$ , since Theorem 1 involves a Gauss sum, which lies in  $\mathbf{Z}[\zeta_{q-1}, \zeta_p]$  but not usually in  $\mathbf{Z}[\zeta_{q-1}]$ .

In Cases 1 and 2, by (\*) we want to prove  $\text{ord}_{\mathfrak{P}}(J(\omega_p^{-k_1}, \dots, \omega_p^{-k_r})) > 0$ . By both Stickelberger's congruence and Lemma 3,

$$\text{ord}_{\mathfrak{P}}(J(\omega_p^{-k_1}, \dots, \omega_p^{-k_r})) = \begin{cases} s_q(k_1) + \cdots + s_q(k_r) - s_q(k_1 + \cdots + k_r) \\ \text{in Case 1} \\ s_q(k_1) + \cdots + s_q(k_r) - f(p-1) \\ \text{in Case 2,} \end{cases}$$

and in both cases the expression on the right is positive by Corollary 1. To prove Cases 3 and 4, note by [11, p. 324] that  $(\zeta_p - 1)^{p-1} = -pu$ , where  $u \equiv 1 \pmod{(\zeta_p - 1)}$ , hence  $u \equiv 1 \pmod{\mathfrak{P}}$ .

In Case 3, Stickelberger's congruence and Lemma 3 yield

$$\begin{aligned} \frac{J(\omega_p^{-k_1}, \dots, \omega_p^{-k_r})}{(\zeta_p - 1)^{s_q(k_1) + \dots + s_q(k_r)}} \cdot q &\equiv \frac{1}{h_q(k_1)} \cdot \dots \cdot \frac{1}{h_q(k_r)} \pmod{\mathfrak{P}} \\ &\equiv \frac{1}{H_p(k_1)} \cdot \dots \cdot \frac{1}{H_p(k_r)} \pmod{\mathfrak{P}} \text{ since } 0 \leq k_i < q-1 \\ &\equiv \frac{(-p)^{\text{ord}_p(k_1!) + \dots + \text{ord}_p(k_r!)}}{k_1! \cdot \dots \cdot k_r!} \pmod{\mathfrak{P}}. \end{aligned}$$

Since  $s_q(k_i) = S_p(k_i)$ ,

$$\begin{aligned} (\zeta_p - 1)^{s_q(k_1) + \dots + s_q(k_r)} &= (\zeta_p - 1)^{k_1 + \dots + k_r - (p-1)(\text{ord}_p(k_1!) + \dots + \text{ord}_p(k_r!))} \\ &= (\zeta_p - 1)^{(p-1) \left( \frac{q-1}{p-1} - (\text{ord}_p(k_1!) + \dots + \text{ord}_p(k_r!)) \right)} \\ &= (-pu)^{\frac{q-1}{p-1} - \text{ord}_p(k_1! \cdot \dots \cdot k_r!)}. \end{aligned}$$

So

$$\frac{J(\omega_p^{-k_1}, \dots, \omega_p^{-k_r}) q (-pu)^{\text{ord}_p(k_1! \cdot \dots \cdot k_r!)}}{(-pu)^{\frac{q-1}{p-1}}} \equiv \frac{(-p)^{\text{ord}_p(k_1! \cdot \dots \cdot k_r!)}}{k_1! \cdot \dots \cdot k_r!} \pmod{\mathfrak{P}},$$

which implies by the congruence  $u \equiv 1 \pmod{\mathfrak{P}}$  and by multiplication by  $(q-1)! = (k_1 + \dots + k_r)!$  that

$$\begin{aligned} J(\omega_p^{-k_1}, \dots, \omega_p^{-k_r}) \frac{q!}{(-p)^{\frac{q-1}{p-1}}} (-p)^{\text{ord}_p(k_1! \cdot \dots \cdot k_r!)} &\equiv \\ \frac{(k_1 + \dots + k_r)!}{k_1! \cdot \dots \cdot k_r!} (-p)^{\text{ord}_p(k_1! \cdot \dots \cdot k_r!)} &\pmod{\mathfrak{P}^{1 + (p-1)\text{ord}_p((q-1)!)}}. \end{aligned}$$

Since

$$\begin{aligned} &1 + (p-1)\text{ord}_p((q-1)!) - (p-1)\text{ord}_p(k_1! \cdot \dots \cdot k_r!) \\ &= 1 + q - 1 - S_p(q-1) - k_1 - \dots - k_r + S_p(k_1) + \dots + S_p(k_r) \\ &= 1 - f(p-1) + s_q(k_1) + \dots + s_q(k_r) \text{ since } 0 \leq k_i < q-1 \\ &\geq 1 \text{ by Corollary 1,} \end{aligned}$$

we see

$$J(\omega_p^{-k_1}, \dots, \omega_p^{-k_r}) \cdot \frac{q!}{(-p)^{\frac{q-1}{p-1}}} \equiv \frac{(k_1 + \dots + k_r)!}{k_1! \cdot \dots \cdot k_r!} \pmod{\mathfrak{P}},$$

so the congruence

$$\frac{q!}{(-p)^{\frac{q-1}{p-1}}} = \frac{q!}{(-p)^{\text{ord}_p(q!)}} \equiv H_p(q) = 1 \pmod{p}$$

settles Case 3.

Finally, in Case 4, Stickelberger's congruence and Lemma 3 imply that

$$\frac{J(\omega_p^{-k_1}, \dots, \omega_p^{-k_r})}{(\zeta_p - 1)^{s_q(k_1) + \dots + s_q(k_r) - s_q(k_1 + \dots + k_r)}} \equiv \frac{h_q(k_1 + \dots + k_r)}{h_q(k_1) \cdot \dots \cdot h_q(k_r)} \pmod{\mathfrak{P}},$$

so

$$\frac{J(\omega_p^{-k_1}, \dots, \omega_p^{-k_r})}{(\zeta_p - 1)^{(p-1)\text{ord}_p\left(\frac{(k_1 + \dots + k_r)!}{k_1! \cdot \dots \cdot k_r!}\right)}} \equiv \frac{(k_1 + \dots + k_r)!}{k_1! \cdot \dots \cdot k_r!} \cdot \frac{1}{(-p)^{\text{ord}_p\left(\frac{(k_1 + \dots + k_r)!}{k_1! \cdot \dots \cdot k_r!}\right)}} \pmod{\mathfrak{P}},$$

since  $s_q(k_i) = S_p(k_i)$  and  $s_q(k_1 + \dots + k_r) = S_p(k_1 + \dots + k_r)$ . Thus

$$J(\omega_p^{-k_1}, \dots, \omega_p^{-k_r}) \equiv \frac{(k_1 + \dots + k_r)!}{k_1! \cdot \dots \cdot k_r!} \pmod{\mathfrak{P}}. \quad \square$$

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