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Autor(en): **Reeder, Mark**

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ON THE COHOMOLOGY OF COMPACT LIE GROUPS

by Mark Reeder

Abstract. We give a new computation of the cohomology of a Lie group that some mathematicians may find to be shorter and more elementary than previous approaches. The main new ingredient is a result of L. Solomon on differential forms invariant under a finite reflection group. The cohomology is shown to have a bi-grading which has several interpretations.

1. Introduction

Let G be a compact connected Lie group, and let T be a maximal torus in G. We denote the corresponding Lie algebras by $\mathfrak{g}$ and $\mathfrak{t}$. Let W be the Weyl group of T in G. Then W acts on $\mathfrak{t}$ as a group generated by reflections about the kernels of the roots of $\mathfrak{t}$ in $\mathfrak{g} \otimes \mathbf{C}$. It has been known since the first half of this century that the cohomology ring $H(G)$, with real coefficients, is an exterior algebra with generators in degrees $2m_1 + 1, \ldots, 2m_l + 1$, where $m_1 + 1, \ldots, m_l + 1$ are the degrees of homogeneous generators of the ring of W-invariant polynomial functions on $\mathfrak{t}$. In particular, the Poincaré polynomial of G is $(1 + t^{2m_1+1}) \cdots (1 + t^{2m_l+1})$, and G has the cohomology of a product of odd-dimensional spheres.

Despite its age and familiarity, it is not easy to find a proof of this theorem in the literature. There are many beginnings and sketches in the textbooks, but the difficult part, namely the remarkable connection between degrees of invariant polynomials and Betti numbers, usually goes unproven. One reason is that the standard proofs (for example, [Bo2], [Ch], [L]) require substantial algebraic preliminaries on Hopf algebras, spectral sequences, and differential algebras. (See [Bo1] and [Sam] for historical surveys.)

We offer here a new but less sophisticated computation of the cohomology of a Lie group, avoiding the above algebraic techniques. Instead we use

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standard Lie theory and more invariant theory than is customary. We have tried to give a fairly complete treatment, in which it is seen that only a few ideas are used repeatedly. The most serious omission of proof is that of Chevalley's theorem on invariant polynomials, but that has many accessible references. Hopefully, enough background has been included to make the whole story coherent to someone with a basic knowledge of Lie groups and differential topology.

A sketch of our computation of $H(G)$ goes as follows. Consider the manifold M consisting of pairs (g, T'), where $g \in G$ and T' is a maximal torus in G which contains g. There is a natural map $\psi: M \to G$, known already to Weyl, given by conjugation. We make the apparently new observation that ψ induces an isomorphism of real cohomology rings $H(M) \simeq H(G)$. This uses only standard facts gleaned from the differential of ψ. One could also invoke the spectral sequence of the fibration $G \to G/T$. This spectral sequence was shown by Leray to degenerate at E^3. It in fact has a spectral subsequence (the W-invariants) which already degenerates at E^2 and still computes $H(G)$ (see (6.4) below).

We still have to compute the cohomology of M. It is easy to see that $H(M) = [H(G/T) \otimes H(T)]^W$. The ring $H(T)$ is naturally isomorphic to the exterior algebra of $\mathfrak{t}^*$, and $H(G/T)$ is isomorphic to the space $\mathscr{H}$ of W-harmonic polynomials on $\mathfrak{t}$, according to a famous theorem of Borel. For completeness, we give a proof of this in the same elementary, if less efficient spirit. As with all proofs, the essential thing is to show that the odd cohomology of G/T vanishes. We do this with a direct generalization of the Morse-theoretic computation of the cohomology of the two-sphere.

So now we are down to invariant theory, and must compute $[\mathscr{H} \otimes \Lambda\mathfrak{t}^*]^W$. This follows immediately from Solomon's determination of the W-invariant differential forms on $\mathfrak{t}$ with polynomial coefficients, which in turn depends on Chevalley's well-known description of W-invariant polynomials. This gives us the desired connection between degrees of W-invariants and Betti numbers of G. Solomon's result also leads to pretty formulas for the multiplicities of the W-modules $\Lambda^q\mathfrak{t}^*$ in spaces of harmonic polynomials (see (3.8)), as well as a generalization of a classical result on the Jacobian of the basic invariants (see (3.9)).

The paper is organized as follows: First the structure of G and its adjoint representation is recalled, then comes invariant theory, followed by the proof of Borel's theorem, finishing with the computation of $H(G)$ and some remarks on its natural bigrading. Throughout, cohomology has real coefficients.

2. Basic material

For more details in this section see [A], for example.

(2.1) Recall that G is a compact connected Lie group with maximal torus T, having respective Lie algebras $\mathfrak{g}$ and $\mathfrak{t}$. The Weyl group is the finite group $W = N/T$, where N is the normalizer in G of T. Since G is compact, there is an $Ad(G)$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$, obtained by averaging any inner product over G. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$ with respect to this inner product, so

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m} \qquad \text{(orthogonal)} .$$

The infinitesimal version of invariance of the inner product is the identity

$$\langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle = 0 ,$$

for $X, Y, Z \in \mathfrak{g}$.

(2.2) The exponential map $\exp : \mathfrak{g} \to G$ is surjective, since G is compact. This is one of the deeper theorems in a first course on Lie groups. We actually only need this surjectivity for $\exp : \mathfrak{t} \to T$, which is clear.

The Lie algebra $\mathfrak{t}$ is abelian (the bracket is zero); in fact $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{g}$. In particular, no nonzero vector in $\mathfrak{m}$ has zero bracket with all of $\mathfrak{t}$. Likewise, $Ad(T)$ has no nonzero invariant vectors in $\mathfrak{m}$.

Now a torus is a topologically cyclic group. That means there exists a *generic element* $t_0 \in T$ whose powers form a dense subgroup of T. It follows that the single operator $Ad(t_0)$ can have no invariants in $\mathfrak{m}$. Likewise in the group G, it can be shown that a maximal torus is its own centralizer, so the centralizer in G of t_0 is just T. There is a similar notion in the Lie algebra. A *regular element* of $\mathfrak{t}$ is one whose $Ad(G)$-centralizer is exactly $Ad(T)$. To find one, take any $H_0 \in \mathfrak{t}$ such that $\exp H_0 = t_0$.

(2.3) The group G acts on $\mathfrak{g}$ via Ad, and this induces an action of W on $\mathfrak{t}$. No element of W acts trivially, and the image of W in $GL(\mathfrak{t})$ is generated by reflections about certain hyperplanes defined as follows.

Since the nontrivial irreducible representations of a torus are given by two dimensional rotations, we have an orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_\nu$, where each $\mathfrak{m}_k$ is two dimensional and there is a finite set of nonzero linear functionals $\Delta^+ = \{\alpha_1, \ldots, \alpha_\nu\} \subset \mathfrak{t}^*$, called *positive roots* such that for $H \in \mathfrak{t}$, the eigenvalues of $Ad \exp H$ on $\mathfrak{m}_i$ are $\exp(\pm\sqrt{-1}\,\alpha_i(H))$. We determine the signs as follows. Fix a regular element $H_0 \in \mathfrak{t}$. We may and shall choose the positive roots so that they take strictly positive values on H_0. The action of W on $\mathfrak{t}$ is generated by reflections about the kernels of the positive roots.

Since each $\mathfrak{m}_i$ is also preserved by $ad(\mathfrak{t})$, we can choose an orthonormal basis $\{X_i, X_{\nu+i}\}$ of $\mathfrak{m}_i$ such that, for $H \in \mathfrak{t}$, the matrix of $ad(H)|_{\mathfrak{m}_i}$ with respect to this basis is

$$\begin{pmatrix} 0 & \alpha(H) \\ -\alpha(H) & 0 \end{pmatrix}.$$

Note that the ad-invariance of the inner product $\langle\ ,\ \rangle$ implies, for all $1 \leqslant i \leqslant \nu$, all $1 \leqslant j \leqslant 2\nu$ and all $H \in \mathfrak{t}$ that

$$\langle H, [X_i, X_j] \rangle = \langle [H, X_i], X_j \rangle = -\alpha_i(H) \langle X_{i+\nu}, X_j \rangle .$$

By orthonormality, this last pairing can only be nontrivial if $j = i + \nu$. Hence if $j \neq i + \nu$, we have $[X_i, X_j] \in \mathfrak{m}$. The same thing happens if $i > \nu$ and $j \neq i - \nu$.

On the other hand, for $1 \leqslant i \leqslant \nu$, set $H_i = [X_i, X_{\nu+i}]$. This is $Ad(T)$-invariant, so $H_i \in \mathfrak{t}$, and $ad(H_i)\mathfrak{m}_i \subseteq \mathfrak{m}_i$. It follows that the span of $X_i, X_{i+\nu}, H_i$ is a Lie subalgebra $\mathfrak{g}_i$ of $\mathfrak{g}$. It is always isomorphic to $\mathfrak{su}(2)$.

3. Invariant Theory

All proofs missing from this section may be found in the textbook [H], the expository article [F], or [Bk].

(3.1) Let

$$\mathcal{S} = \bigoplus_{p=0}^{\infty} \mathcal{S}^p \quad \text{and} \quad \Lambda = \bigoplus_{q=0}^{l} \quad (l = \dim \mathfrak{t})$$

be the symmetric and exterior algebras on $\mathfrak{t}^*$, respectively. The adjoint action of W on $\mathfrak{t}$ induces representations of W on $\mathcal{S}$ and Λ by degree-preserving algebra automorphisms. For example, the action of W on Λ^l is multiplication by the *sign character*

$$\varepsilon : W \to \{\pm 1\} \quad \text{given by} \quad \varepsilon(w) = \det Ad(w)_{\mathfrak{t}} .$$

Note that $\varepsilon(w)$ is the parity of the number of reflections needed to express $Ad(w)_{\mathfrak{t}}$.

We are interested in W-invariant polynomials, and more generally, W-invariant differential forms with polynomial coefficients. For the unitary group $U(n)$, the ring of invariants $\mathscr{S}^W$ is generated by the elementary symmetric polynomials $s_1, \ldots, s_n$ in variables $x_1, \ldots, x_n$ defined as

$$s_d(x_1, \ldots, x_n) = \sum_{1 \leqslant i_1 < \cdots < i_d \leqslant n} x_{i_1} \cdots x_{i_d} .$$

The elementary symmetric polynomials are algebraically independent, and their number equals the dimension n of a maximal torus of $U(n)$. In general, we have

(3.2) THEOREM (Chevalley). *The ring* $\mathscr{S}^W$ *has algebraically independent homogeneous generators* $F_1, \ldots, F_l$, *hence is a polynomial ring*

$$\mathscr{S}^W = \mathbf{R}[F_1, \ldots, F_l] .$$

We number these generators so that $\deg F_1 \leqslant \deg F_2 \leqslant \cdots \leqslant \deg F_l$. (Note to experts: Since we are not assuming G to be semisimple, some of the F_i's could have degree one.) The *exponents* $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_l$ of W acting on $\mathfrak{t}$ are defined by the relations $m_i + 1 = \deg F_i$. It is known that $m_1 + \cdots + m_l = \nu$, and $(1 + m_1) \cdots (1 + m_l) = |W|$.

Every compact connected Lie group is, up to finite covering, the product of a central torus with a direct product of classical groups $SU(n)$, $SO(n)$, $Sp(n)$, and exceptional groups G_2, F_4, E_6, E_7, E_8. For these groups the m_i's are given as follows:

$SU(n)$: $1, 2, \ldots, n-1$. $\qquad SO(2n)$: $1, 3, \ldots, 2n-3, n-1$.

$SO(2n+1)$ and $Sp(n)$: $1, 3, \ldots, 2n-1$.

G_2: $1, 5$. $\qquad F_4$: $1, 5, 7, 11$.

E_6: $1, 4, 5, 7, 8, 11$.

E_7: $1, 5, 7, 9, 11, 13, 17$.

E_8: $1, 7, 11, 13, 17, 19, 23, 29$.

These numbers are easy to verify for the classical groups and G_2 (whose maximal torus T is that of $SU(3)$ with Weyl group S_3 extended by the inverse map on T), using elementary symmetric polynomials as above. Computing the exponents for the other exceptional groups is more difficult. See [C].

(3.3) The W-module structure of the whole polynomial ring $\mathscr{S}$ is given as follows. Let $\mathscr{D}$ be the ring of constant coefficient differential operators on $\mathscr{S}$. We can think of $\mathscr{D}$ as the symmetric algebra $S(\mathfrak{t})$, where $H \in \mathfrak{t}$

corresponds to the derivation of $\mathcal{S}$ extending the functional on $\mathfrak{t}^*$ given by evaluation at H. Then W acts naturally on $\mathcal{D}$ and one defines the "harmonic polynomials" in $\mathcal{S}$ to be those annihilated by the W-invariant differential operators:

$$\mathcal{H} = \{f \in \mathcal{S}: \mathcal{D}^W f = 0\} .$$

Let $\mathcal{H}^p = \mathcal{H} \cap \mathcal{S}^p$. Then $\mathcal{H} = \oplus_p \mathcal{H}^p$, since a differential operator is W-invariant only if each of its homogeneous components is so. The action of W on $\mathcal{S}$ leaves $\mathcal{H}$ invariant.

Let $\mathcal{I}$ be the ideal in $\mathcal{S}$ generated by the elements of $\mathcal{S}^W$ of positive degree. It is known (see [H, p. 360] that $\mathcal{S} = \mathcal{H} \oplus \mathcal{I}$, and the multiplication map is a linear isomorphism $\mathcal{H} \otimes \mathcal{S}^W \xrightarrow{\simeq} \mathcal{S}$. The former implies that $\mathcal{S}/\mathcal{I}$ and $\mathcal{H}$ are isomorphic W-modules. They are in fact isomorphic to the regular representation of W, as we shall see in (5.4). The isomorphism $\mathcal{H} \otimes \mathcal{S}^W \simeq \mathcal{S}$ implies the identity

$$\sum_{p \geqslant 0} \dim \mathcal{H}^p t^p = \prod_{i=1}^{l} (1 + t + t^2 + \cdots + t^{m_i}) ,$$

which in turn shows that $\dim \mathcal{H}^\nu = 1$, and $\mathcal{H}^p = 0$ for $p > \nu$.

(3.4) Let V be any irreducible W-module. Suppose V is a constituent of $\mathcal{S}^b$, and not a constituent of $\mathcal{S}^c$, for any $c < b$. We call b the *birthday* of V. Then the V-isotypic component of $\mathcal{S}^b$ must consist of harmonic polynomials, for otherwise, a W-invariant differential operator of positive degree would intertwine V with a space of polynomials of lower degree.

For example, the *primordial* harmonic polynomial is

$$\Pi = \prod_{\alpha \in \Delta^+} \alpha \in \mathcal{H}^\nu ,$$

where we recall that Δ^+ is the set of positive roots. For $U(n)$, Π is the van der Monde determinant $\prod_{i<j} x_i - x_j$, which transforms under the symmetric group S_n by the sign character. In general, Π transforms by the sign character ε of W, and any other polynomial transforming by ε must vanish on all root hyperplanes, hence be divisible by Π. Therefore Π is harmonic, ν is the birthday of ε and (1.4) shows that $\mathcal{H}^\nu$ is spanned by Π.

We say that Π is primordial because $\mathcal{H}$ is spanned by the partial derivatives of Π (see [S]). This turns out to be the algebraic analogue of Poincaré duality for G/T.

As we have seen, the sign character is also afforded by Λ^l. In general, if $\mathfrak{g}$ is simple then each exterior power Λ^q is an irreducible W-module. We shall determine the birthday of each Λ^q shortly.

(3.5) Now consider the algebra $\mathscr{S} \otimes \Lambda$ of differential forms on $\mathfrak{t}$ with polynomial coefficients. Let $F_1, \ldots, F_l$ be homogeneous generators of $\mathscr{S}^W$ as in (3.2). Extending that result, Solomon [Sol] has described the W-invariants in $\mathscr{S} \otimes \Lambda$. Because it seems not so well known but is important here, we give a proof, taken from [H].

(3.6) THEOREM (Solomon). *The space* $(\mathscr{S} \otimes \Lambda)^W$ *of* W-*invariants in* $\mathscr{S} \otimes \Lambda$ *is a free* $\mathscr{S}^W$-*module with basis*

$$\{dF_{i_1} \wedge \cdots \wedge dF_{i_q} : 1 \leqslant i_1 < \cdots < i_q \leqslant l\}\,.$$

Proof. It is a general fact about polynomials that the algebraic independence of $F_1, \ldots, F_l$ is equivalent to the form $dF_1 \wedge \cdots \wedge dF_l$ not being identically zero. Let $x_1, \ldots, x_l$ be a basis of $\mathfrak{t}^*$. Then

$$dF_1 \wedge \cdots \wedge dF_l = J dx_1 \cdots dx_l\,,$$

where the Jacobian J is a polynomial of degree $m_1 + \cdots + m_l = \nu$. The left side is W-invariant and $dx_1 \wedge \cdots \wedge dx_l$ affords the sign character ε. Hence J must also afford ε and, because of its degree, J must be a nonzero multiple of the primordial harmonic polynomial Π. Thus

$$dF_1 \wedge \cdots \wedge dF_l = c\Pi dx_1 \wedge \cdots \wedge dx_l\,,$$

for some nonzero real number c.

For a sequence $I = i_1 < \cdots < i_q$, let I' be the increasing sequence of all integers in $\{1, \ldots, l\} - \{i_1, \ldots, i_q\}$. Set $dF_I = dF_{i_1} \wedge \cdots \wedge dF_{i_q}$ for any sequence I. Let k be the quotient field of $\mathscr{S}$. If $f_I \in k$ are such that $\sum_I f_I dF_I = 0$ then multiplying by $dF_{I'}$ kills all terms but I, leaving $\pm c f_I \Pi dx_1 \cdots dx_l = 0$, so $f_I = 0$. Counting dimensions, we find that the dF_I are a k-basis of $k \otimes \Lambda$, and are in particular linearly independent over $\mathscr{S}^W$. Now suppose $\omega \in \mathscr{S} \otimes \Lambda$ is W-invariant. We can express $\omega = \sum_I g_I dF_I$ for some $g_I \in k$. Multiplying by $dF_{I'}$ again, we have

$$\omega \wedge dF_{I'} = \pm c g_I \Pi dx_1 \cdots dx_l \in [\mathscr{S} \otimes \Lambda]^W\,.$$

This forces g_I to be not only W-invariant, but also polynomial. $\square$

For $\omega \in \mathscr{S} \otimes \Lambda$, let $\omega' \in \mathscr{S}/\mathscr{I} \otimes \Lambda$ be obtained by reducing the coefficients of ω modulo $\mathscr{I}$. This induces an exact sequence

$$0 \to (\mathscr{I} \otimes \Lambda)^W \to (\mathscr{S} \otimes \Lambda)^W \xrightarrow{\omega \mapsto \omega'} (\mathscr{S}/\mathscr{I} \otimes \Lambda)^W \to 0\,.$$

It follows immediately from Solomon's theorem that $\{dF'_{i_1} \wedge \cdots \wedge dF'_{i_q} : 1 \leqslant i_1 < \cdots < i_q \leqslant l\}$ spans $(\mathscr{S}/\mathscr{I} \otimes \Lambda)^W$ (over $\mathbf{R}$). This is in fact a basis, since $\mathcal{S}/\mathcal{I}$ affords the regular representation of W, so $\dim(\mathcal{S}/\mathcal{I} \otimes \Lambda)^W = 2^l$. We therefore have the following

(3.7) COROLLARY. *$(\mathcal{S}/\mathcal{I} \otimes \Lambda)^W$ is an exterior algebra with generators*

$$dF_i' \in [(\mathcal{S}/\mathcal{I})^{m_i} \otimes \Lambda^1]^W, \quad \textit{for } 1 \leqslant i \leqslant l .$$

We will see later that this exterior algebra is, with degrees in $\mathcal{S}/\mathcal{I}$ doubled, the cohomology ring of the compact Lie group G. As W-representations, we have $\mathcal{S}/\mathcal{I} \simeq \mathcal{H}$ and the corollary gives the following

(3.8) MULTIPLICITY FORMULA.

$$\sum_{n=0}^{\vee} \dim \operatorname{Hom}_W(\Lambda^q, \mathcal{H}^n) u^n = s_q(u^{m_1}, \ldots, u^{m_l}) ,$$

where s_q is the elementary symmetric polynomial in l-variables, and the m_i are the exponents of W.

In particular, the birthday of Λ^q is $m_1 + \cdots + m_q$, if $\mathfrak{g}$ is simple.

(3.9) We close this section with a digression. Suppose $\mathfrak{g}$ is simple, so all Λ^q are irreducible W-modules. We can actually witness the birth of Λ^q in $\mathcal{H}$ using the differentials dF_i, as follows. Choose a basis $x_1, \ldots, x_l$ of $\mathfrak{t}^*$, and consider a q-form

$$\omega = \sum f_{i_1, \ldots, i_q} dx_{i_1} \wedge \cdots \wedge dx_{i_q} \in \mathcal{S} \otimes \Lambda^q .$$

The linear span of the coefficient polynomials $f_{i_1, \ldots, i_q}$ is independent of the choice of basis $\{x_i\}$. Moreover, if ω is W-invariant and nonzero, then its coefficients span a W-invariant subspace of $\mathcal{S}$ which is isomorphic to Λ^q as a W-module, since the latter is irreducible and self-contragredient.

For example, we have seen that

$$dF_1 \wedge \cdots \wedge dF_l = c \Pi dx_1 \wedge \cdots \wedge dx_q ,$$

where c is a nonzero scalar, and Π is the primordial harmonic polynomial, affording the sign character of W. We have a generalization of this for all Λ^q.

(3.10) PROPOSITION. *For $1 \leqslant q \leqslant l$, the coefficients of $dF_1 \wedge \cdots \wedge dF_q$ are harmonic polynomials. They span an irreducible W-submodule of $\mathcal{H}^{m_1 + \cdots + m_q}$, isomorphic to Λ^q.*

Proof. The coefficients of $dF_1 \wedge \cdots \wedge dF_q \in (S^{m_1 + \cdots + m_q} \otimes \Lambda^q)^W$ span a W-invariant subspace of $S^{m_1 + \cdots + m_q}$, isomorphic to Λ^q. As in (3.4), these coefficients are harmonic because $m_1 + \cdots + m_q$ is the birthday of Λ^q, by the multiplicity formula (3.8). $\square$

4. Invariant Differential Forms

The ideas in this section go back to E. Cartan and de Rham. For a thorough exposition, see [C-E].

(4.1) Suppose a compact Lie group G acts transitively on a manifold M. Let τ_g be the diffeomorphism of M corresponding to $g \in G$. A differential p-form $\omega \in \Omega^p(M)$ is *G-invariant* if $\tau_g^* \omega = \omega$. Such a form is determined by its value at any one point of M. One shows by averaging that every de Rham cohomology class on M is represented by a G-invariant form, and that the subcomplex of invariant forms is preserved by the exterior derivative.

Identify $M = G/K$ where K is the stabilizer of a point $o \in M$. We have an orthogonal decomposition $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{n}$, where $\mathfrak{r}$ is the Lie algebra of K. Moreover this decomposition is preserved by $Ad(K)$. For example if G acts on itself by left multiplication then $K = 1$ and $\mathfrak{n} = \mathfrak{g}$. For another example take $M = G/T$, so $K = T$ and $\mathfrak{n} = \mathfrak{m}$. In general, $\mathfrak{n}$ is naturally identified with the tangent space $T_o(M)$, so an invariant form $\tilde{\omega}$ is determined by the skew-symmetric multilinear map

$$\omega = \tilde{\omega}_o : \mathfrak{n} \times \cdots \times \mathfrak{n} \to \mathbf{R} .$$

That is, $\omega \in \Lambda^p \mathfrak{n}^*$. The invariance of $\tilde{\omega}$ under K implies the $Ad(K)$-invariance of ω. Conversely, any element $\omega \in (\Lambda^p \mathfrak{n}^*)^K$ determines a G-invariant form $\tilde{\omega}$ on M by the formula

$$\tilde{\omega}_{g \cdot o}\big((d\tau_g)_o X_1, \ldots, (d\tau_g)_o X_p\big) = \omega(X_1, \ldots, X_p) ,$$

for $X_1, \ldots, X_p \in \mathfrak{n}$ and $g \in G$. Thus we may identify the G-invariant p-forms on M with the space $(\Lambda^p \mathfrak{n}^*)^K$. In this view, the exterior derivative becomes the map $\delta : (\Lambda^p \mathfrak{n}^*)^K \to (\Lambda^{p+1} \mathfrak{n}^*)^K$ given by

$$\delta\omega(X_0, \ldots, X_p) = \frac{1}{p+1} \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j]_{\mathfrak{n}}, X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p) .$$

Here $\hat{}$ means the term is omitted, and $[X_i, X_j]_{\mathfrak{n}}$ is the projection of $[X_i, X_j]$ into $\mathfrak{n}$ along $\mathfrak{r}$. The complex $\{(\Lambda^p \mathfrak{n}^*)^K, \delta\}$ computes the de Rham cohomology of M.

(4.2) Assume that K is connected. Since any homomorphism from K to the multiplicative real numbers must be trivial, the determinant is a nonzero element of the one dimensional space $(\Lambda^n \mathfrak{n}^*)^K$, where $n = \dim M$. It follows that M is orientable, and any G-invariant n-form on M will have nonzero integral over M as soon as it does not vanish at one point.

(4.3) In the case $M = G$ we have the additional symmetry of left and right multiplication by $G \times G$, and every cohomology class contains a bi-invariant representive. The value at e of a bi-invariant form is $Ad(G)$ invariant. Taking the derivative of the condition for $\omega \in \Lambda^p \mathfrak{g}^*$ to be $Ad(G)$-invariant, we find (product rule) that $\omega([X, X_1], X_2, \ldots, X_p) + \cdots + \omega(X_1, \ldots, [X, X_p]) = 0$ for all $X, X_1, \ldots, X_p \in \mathfrak{g}$. It is then not hard to show that this condition implies that $\delta\omega = 0$. Hence all bi-invariant forms are closed. Since δ commutes with Ad, it follows that the de Rham cohomology of G is computed by the complex $(\Lambda \mathfrak{g}^*)^G$, with zero differential. That is, $H(G) \simeq (\Lambda \mathfrak{g}^*)^G$, as graded rings.

5. THE COHOMOLOGY OF FLAG MANIFOLDS

The *Bruhat Decomposition* is a cell decomposition of the flag manifold G/T into even dimensional cells indexed by elements of the Weyl group W. It generalizes the decomposition of the two-sphere (flag manifold of $SU(2)$) into a point and an open disk. The existence of such a decomposition implies that there are no boundary maps in cellular homology, and the cohomology of $H(G/T)$ is nonzero only in even degrees.

It is customary to explain the The Bruhat decomposition in terms of complex groups. For example the flag manifold for $U(n)$ is in fact a homogeneous space for $GL_n(\mathbf{C})$, and the cells can be described as the orbits of certain subgroups of the group of upper triangular complex matrices, which do not live in $U(n)$. We shall, however, describe the cell decomposition of G/T purely in terms of the compact group G, using Morse theory. It was Bott, later with Samelson, who first applied Morse theory to the loop space of G from which, combined with results of Borel and Leray, they deduced results on the topology of G and G/T. See [BT] for a brief introduction to Morse theory.

(5.1) We need to find a "Morse function" f on G/T. This is a smooth real valued function whose Hessian (matrix of second partial derivatives taken in local coordinates) at each critical point has nonzero determinant. How shall we find one? For the unit sphere in $\mathbf{R}^3$ centered at $(0, 0, 0)$, we can take

$f(x, y, z) = z$, and the critical points are the north and south poles, where the Hessian is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, respectively. The flow lines of the gradient of f emanating from the south pole form a 2-cell, and the north pole is a zero-cell. We can also write f using the dot product on $\mathbf{R}^3$ as $f(p) = p \cdot n$, where n is the position vector of the north pole. Viewing $\mathbf{R}^3$ as the Lie algebra $\mathfrak{su}(2)$, this tells us what to do in general.

As analogue of the north pole, we take $H_0 \in \mathfrak{t}$ to be the regular element defining the positive roots, as in (2.3). Since the $Ad(G)$-centralizer of H_0 is exactly T, we may view $G/T \subset \mathfrak{g}$ as the $Ad(G)$-orbit of H_0 (analogous to $S^2 \subset \mathbf{R}^3$), and we define a function $f: G/T \to \mathbf{R}$ by

$$f(gT) = \langle Ad(g)H_0, H_0 \rangle .$$

For $X \in \mathfrak{g}$, let $\bar{X}$ be the vector field on G/T given, for a smooth function ϕ on G/T, by

$$\bar{X}\phi(gT) = \frac{d}{ds}\phi((\exp sX)\, gT)\,|_{s=0} .$$

Then a short computation, using the ad-invariance of the inner product, shows that

$$\bar{X}f(gT) = \langle Ad(g)H_0, [H_0, X] \rangle .$$

Since the centralizer of H_0 in $\mathfrak{g}$ is exactly $\mathfrak{t}$, the image of $ad(H_0)$ is all of $\mathfrak{m}$. So gT is a critical point of f if and only if $\langle Ad(G)H_0, \mathfrak{m} \rangle = 0$, forcing $Ad(g)H_0 \in \mathfrak{t}$. It then follows that $Ad(g)H_0 = Ad(w)H_0$ for some $w \in W$. So the critical points of f are the wT, for $w \in W$.

Let $X_1, X_2, \ldots, X_{2\nu}$ be the orthonormal basis of $\mathfrak{m}$ from (2.3). For each $w \in W$, the differential of the projection $\pi: G \to G/T$ maps $Ad(w)\mathfrak{m} = \mathfrak{m}$ isomorphically onto $T_{wT}(G/T)$, so we can use the X_i's to compute the Hessian of f at each point wT. Let $h_{ij}(w)$ be the ij entry in the Hessian matrix. Another short computation gives

$$h_{ij}(w) = \bar{X}_i\bar{X}_j f(wT) = \langle [X_i, Ad(w)H_0], [H_0, X_j] \rangle .$$

Recalling the bracket relations in (2.3), we find that

$$h_{ii}(w) = -\alpha_i(Ad(w)H_0)\,\alpha_i(H_0) ,$$

and $h_{ij}(w) = 0$ if $i \neq j$. The regularity of H_0 implies that of $Ad(w)H_0$, so the Hessian is nonsingular. It follows that the index of the critical point wT, by definition the number of negative eigenvalues of the Hessian, is twice the number $m(w)$ of positive roots α such that $w^{-1}\alpha$ is again positive. Now by the main theorem of Morse theory, the Poincaré polynomial of G/T is $\sum_{w \in W} u^{2m(w)}$. In particular, $H^{\text{odd}}(G/T) = 0$ and $\dim H^{\text{even}}(G/T) = \dim H(G/T) = \chi(G/T) = |W|$.

(5.2) The *Schubert cell* X_w in the Bruhat decomposition is spanned by those flow lines of the gradient of f which emanate from wT. The dimension of this cell then equals twice the number of positive eigenvalues of the Hessian at wT, which is the number of positive roots made negative by w.

(5.3) Recalling that W acts on G/T, we use Leray's argument to determine the structure of the W-module $H(G/T)$, ignoring the grading for now. The element $w \in W$ acts by $w \cdot (gT) = gw^{-1}T$ of W on G/T. Since there is no cohomology in odd degrees, the Lefschetz number of w equals its trace on $H(G/T)$. If $w \neq 1$ there are no fixed points so the Lefschetz number is zero. If $w = 1$ we are computing the Euler characteristic which we now know is $|W|$. Hence the trace of any $w \in W$ acting on $H(G/T)$ is that of the regular representation, so $H(G/T) \simeq \mathbf{R}[W]$ (the group ring of W) as W-modules.

The theorem of Borel is a refinement of this, and describes the W-module structure of $H(G/T)$ in each degree. Recall the graded ring $\mathscr{S}$ of polynomial functions on $\mathfrak{t}$ and its ideal $\mathscr{I}$ generated by the W-invariant polynomials of positive degree. Our object is to prove the following

(5.4) THEOREM (Borel). *There is a degree-doubling W-equivariant ring isomorphism*

$$c: \mathscr{S}/\mathscr{I} \to H(G/T).$$

Consequently, $\mathscr{H}_{(2)} \simeq H(G/T)$, where $\mathscr{H}_{(2)}$ is $\mathscr{H}$ with the grading degrees doubled.

Proof. We will describe the cohomology ring of G/T in terms of G-invariant differential forms. For each $\lambda \in \mathfrak{t}^*$, extended to a functional on all of $\mathfrak{g}$ by making it zero on $\mathfrak{m}$, define an $Ad(T)$-invariant two-form ω_λ on $\mathfrak{m}$ by

$$\omega_\lambda(X, Y) = \lambda([X, Y]).$$

As in (4.1), this corresponds to a left-invariant form $\tilde{\omega}_\lambda$ on G/T.

Though it is not needed here, one can show that if λ is the differential of a character $\chi_\lambda: T \to S_1$, then $\frac{1}{4\pi}\omega_\lambda$ represents the first Chern class of the corresponding complex line bundle $G \times_T \mathbf{C}$, where T acts on $\mathbf{C}$ via χ_λ.

Returning to the proof, note that for $w \in W$, acting on $\mathfrak{t}^*$ by $w\lambda(H) = \lambda(Ad(w)^{-1}H)$, and on the space of differential forms $\Omega(G/T)$ via its action on G/T, we have $w^*\omega_\lambda = \omega_{w\lambda}$. Moreover, the Jacobi identity says that $\delta\omega_\lambda = 0$, and we let $c(\lambda) = [\tilde{\omega}_\lambda] \in H^2(G/T)$ be the cohomology class of $\tilde{\omega}_\lambda$. This extends to a degree-doubling ring homomorphism

$$c: \mathscr{S} \to H(G/T)$$

which also preserves the W-action on both sides. Since $H(G/T)$ is the regular representation of W, its W-invariants are one-dimensional and therefore can occur only in $H^0(G/T)$. By the W-equivariance, it follows that the kernel of c contains the ideal $\mathscr{I} \in \mathscr{S}$ generated by W-invariant polynomials of positive degree. Borel's theorem asserts that $\mathscr{I}$ is exactly the kernel of c.

Since $\mathscr{S} = \mathscr{H} \oplus \mathscr{I}$ (see (3.4)), we shall prove the theorem by showing that the restriction of c to $\mathscr{H}$ is injective. We start in the top dimension, where our task is to show that $c(\Pi)$ (recall from (3.5) that Π is the primordial harmonic polynomial) is nonzero in $H^{2\nu}(G/T)$. One way is to use the Chern class interpretation to show that $c(\Pi)$ is a nonzero multiple of the Euler class of G/T, whose integral over G/T is $\chi(G/T) = |W| \neq 0$. However, we shall be more pedestrian about it, and evaluate $c(\Pi)$ on a basis on $\mathfrak{m}$ (see (4.2)).

Recall that for each positive root α_i, we have elements $X_i, X_{i+\nu}$ in $\mathfrak{m}$, with bracket relations $[X_i, X_{i+\nu}] = H_i \in \mathfrak{t}$, $[X_i, X_j] \in \mathfrak{m}$ if $j \neq i + \nu$. The set $\{X_i : 1 \leqslant i \leqslant 2\nu\}$ is a basis of $\mathfrak{m}$. Write ω_i for ω_{α_i}, so $c(\Pi) = [\tilde{\omega}_1 \wedge \cdots \wedge \tilde{\omega}_\nu]$. We compute

$$\begin{aligned} &\omega_1 \wedge \cdots \wedge \omega_\nu (X_1, X_{1+\nu} \dots, X_\nu, X_{2\nu}) \\ &= \frac{1}{(2\nu)!} \sum_{\sigma \in S_{2\nu}} sgn(\sigma)\omega_1(X_{\sigma(1)}, X_{\sigma(1+\nu)}) \cdots \omega_\nu(X_{\sigma(\nu)}, X_{\sigma(2\nu)}) \\ &= \frac{1}{(2\nu)!} \sum_{\sigma \in S_{2\nu}} sgn(\sigma)\alpha_1([X_{\sigma(1)}, X_{\sigma(1+\nu)}]) \cdots \alpha_\nu([X_{\sigma(\nu)}, X_{\sigma(2\nu)}]) \,. \end{aligned}$$

Now $\alpha_i([X_{\sigma(i)}, X_{\sigma(i+\nu)}]) = 0$ unless $[X_{\sigma(i)}, X_{\sigma(i+\nu)}] \in \mathfrak{t}$, so the σ^{th} term is nonzero only if σ permutes the pairs $\pi_i = \{i, i+\nu\}$ and possibly switches some of the members of each pair. Moreover, $sgn(\sigma)$ equals minus one to the number of switches, so we get

$$\begin{aligned} &\omega_1 \wedge \cdots \wedge \omega_\nu (X_1, X_{1+\nu} \dots, X_\nu, X_{2\nu}) \\ &= \frac{2^\nu}{(2\nu)!} \sum_{\sigma \in S_\nu} \alpha_1([X_{\sigma(1)}, X_{\nu+\sigma(1)}]) \cdots \alpha_\nu([X_{\sigma(\nu)}, X_{\nu+\sigma(\nu)}]) \\ &= \frac{2^\nu}{(2\nu)!} \sum_{\sigma \in S_\nu} \alpha_1(H_{\sigma(1)}) \cdots \alpha_\nu(H_{\sigma(\nu)}) \\ &= \frac{2^\nu}{(2\nu)!} \partial_1 \cdots \partial_\nu \Pi \,, \end{aligned}$$

where, as in (3.3), ∂_i is the derivation of $\mathscr{S}$ extending the functional $\lambda \mapsto \lambda(H_i)$. We have a perfect pairing

$$\mathscr{D} \otimes \mathscr{S} \to \mathbf{R}$$

given by $(D, f) = (Df)(0)$. Since the pairing is perfect, something in degree ν must pair nontrivially with Π. Since an irreducible W-module can only pair nontrivially with its dual, and the self-dual character ε occurs with multiplicity one in $\mathscr{D}^\nu$, afforded by $\partial_1 \cdots \partial_\nu$, we must have $\partial_1 \cdots \partial_\nu \Pi \neq 0$, so $c(\Pi) \neq 0$.

Observe that $\partial_1 \cdots \partial_\nu$ is analogous to the fundamental class of G/T, and the pairing is essentially that between homology and cohomology. We further remark that in fact all irreducible representations of W are defined over the rational numbers, hence they are all self dual. This is a consequence of Springer's cohomological construction of W-modules [Sp].

Returning again to our task, we now inductively assume that $c\colon \mathscr{H}^k \to H^{2k}(G/T)$ is injective for $k \leqslant \nu$, and let $V = \mathscr{H}^{k-1} \cap \ker c$. Note that V is preserved by W since c is W-equivariant. The sign character does not occur in $\mathscr{H}^{k-1}$, so there is a positive root α whose corresponding reflection s_α does not act by $-I$ on V. Decompose $V = V_+ \oplus V_-$ according to the eigenspaces of s_α. If $V \neq 0$ then $V_+ \neq 0$, so take $f \in V_+$. Now $c(\alpha f) = c(\alpha) c(f) = 0$, and αf is in degree k, so we must have $\alpha f \in \mathscr{I}$ by the induction hypothesis. Let $h_1, \ldots, h_{|W|}$ be a basis of $\mathscr{H}$ with $h_1, \ldots, h_r$ s_α-skew and the rest s_α invariant. By Chevalley's theorem (3.2), we can write $\alpha f = \sum h_i \sigma_i$ with σ_i W-invariant of positive degree. Since αf is s_α-skew, the sum only goes up to r. Now for $i \leqslant r$, the polynomial h_i must vanish on $\ker \alpha$, hence can be written $h_i = \alpha h_i'$ for some $h_i' \in \mathscr{S}$. But then $f = \sum_{i=1}^r h_i' \sigma_i \in \mathscr{I}$. Since f is supposed to be harmonic, we must have $f = 0$. Hence c is injective on $\mathscr{H}$, and the proof of Borel's theorem is complete. $\square$

6. The cohomology of a Lie group

We now have all the ingredients for our proof. Consider the map $\psi\colon G/T \times T \to G$ given by $\psi(gT, t) = gtg^{-1}$. The Weyl group W acts on T by conjugation and on G/T by $w \cdot gT = gn^{-1}T$, where $w = nT$. Hence W acts on $H(G/T \times T) = H(G/T) \otimes H(T)$. Since $\psi(gn^{-1}T, wtw^{-1}) = \psi(gT, t)$, it follows that the induced map ψ^* on cohomology maps $H(G)$ to $[H(G/T) \otimes H(T)]^W$. Though we prefer to have it in this form, the latter group could be thought of as the cohomology of the quotient of $G/T \times T$

by the action of W, and this quotient has a natural interpretation. As in the introduction, let M be the set of pairs (g, T') where T' is a maximal torus in G containing $g \in G$. The map $G/T \times_W T \to M$ sending $(gT, t) \bmod (W)$ to (gtg^{-1}, gTg^{-1}) is a diffeomorphism.

PROPOSITION (6.1). *The map induced by* ψ *on cohomology is an isomorphism of graded rings*

$$\psi^* : H(G) \xrightarrow{\cong} [H(G/T) \otimes H(T)]^W .$$

Proof. We compute the derivative $(d\psi)_{(gT,t)}$ at a point $(gT, t) \in G/T \times T$. For each point $gT \in G/T$, we identify $\mathfrak{m}$ with the tangent space $T_{gT}(G/T)$ by letting $X \in \mathfrak{m}$ correspond to the initial tangent vector X_{gT} of the path $s \mapsto g(\exp sX)T$ in G/T. Similarly, an element $X \in \mathfrak{g}$ (resp. $H \in \mathfrak{t}$) corresponds to a tangent vector $X_g \in T_g(G)$ (resp. $H_t \in T_t(T)$, for $t \in T$). Then

$$\begin{aligned}
(d\psi)_{gT,t}(X_{gT}, 0) &= \frac{d}{ds} g(\exp sX)t(\exp - sX)g^{-1}\big|_{s=0} \\
&= \frac{d}{ds} gtg^{-1}[\exp sAd(gt^{-1})X]\,[\exp - sAd(g)X]\big|_{s=0} \\
&= \frac{d}{ds} gtg^{-1}[X + sAd(g)\,(Ad(t^{-1}) - 1)X + O(s^2)]\big|_{s=0} \\
&= [Ad(g)\,(Ad(t^{-1}))X]_{gtg^{-1}} .
\end{aligned}$$

Similarly, we find, for $H \in \mathfrak{t}$, that

$$(d\psi)_{gT,t}(0, H_t) = [Ad(g)H]_{gtg^{-1}} .$$

Hence, under the identifications, $(d\psi)_{(gT,t)}$ is the map

$$(Ad(t^{-1}) - I)_{\mathfrak{m}} \oplus I : \mathfrak{m} \oplus \mathfrak{t} \to \mathfrak{m} \oplus \mathfrak{t} = \mathfrak{g} .$$

Here the subscript $\mathfrak{m}$ means we view $Ad(t^{-1}) - I$ as a map from $\mathfrak{m}$ to itself. Now G being compact and connected, we must have $\det Ad(t) = 1$, so

$$\det (d\psi)_{(gT,t)} = \det (I - Ad(t))_{\mathfrak{m}} .$$

(Actually, $\mathfrak{m}$ is always even-dimensional as we have seen, so there is no need to reverse the subtraction).

We compute the degree of ψ by finding a regular value. Let t_0 be a generic element in T, as in (2.3). Consider $\psi^{-1}(t_0) = \{(gT, t) : gtg^{-1} = t_0\}$. It turns out that any two elements of T conjugate in G must be conjugate by an element of W. (In $U(n)$, two diagonal matrices with the same set of eigenvalues must be conjugate by a permutation matrix.) It follows easily then that

$$\psi^{-1}(t_0) = \{(wT, wt_0w^{-1}) : w \in W\} .$$

We next show that ψ preserves orientation at each point in $\psi^{-1}(t_0)$. The eigenvalues of $Ad(t_0)$ in $\mathfrak{m}$ are complex conjugate pairs $z, \bar{z}$, where $|z| = 1, z \neq 1$. Hence $|1 - z||1 - \bar{z}| = 2(1 - Re(z)) > 0$, so $\det(I - Ad(t_0))_{\mathfrak{m}} > 0$.

At this point we know the degree of ψ is $\deg \psi = |W| \neq 0$. By Poincaré duality, any smooth map between compact manifolds of the same dimension is injective on cohomology as soon as it has nonzero degree. Hence $\psi^* : H(G) \to [H(G/T) \times H(T)]^W$ is injective. We finish the proof of (6.1) by showing that both sides have the same dimension.

For this we use, three times, the following basic principle. Let K be a compact group (here K will be G, T or W). Let dk be the left invariant Haar measure on K with total mass one. Let V be a finite dimensional real vector space, and $\rho : K \to GL(V)$ a continuous group homomorphism. Then the space V^K of vectors fixed by all $\rho(k)$, $k \in K$, has dimension

$$\dim V^K = \int_K \operatorname{trace} \rho(k)\, dk .$$

To compute this integral over G, we must exploit further the computation of $d\psi$. Let $\omega_G, \omega_T, \omega_{G/T}$ be the unique invariant (under left translations by G, T, and G respectively) differential forms of top degree whose integral over the respective manifold is one. The the pull-back formula for integration gives

$$\int_G f\omega_G = \frac{1}{\deg \psi} \int_{G/T \times T} f \circ \psi(gT, t)\, |\det(d\psi)_{(gT,t)}|\, \omega_{G/T} \wedge \omega_T ,$$

where the determinant is computed with respect to bases spanning parallelograms of unit volume with respect to the appropriate forms. Taking f to be invariant under conjugation by G, we arrive at the famous Weyl integration formula:

$$\int_G f\omega_G = \frac{1}{|W|} \int_T f(t) \det(I - Ad(t))_{\mathfrak{m}}\, \omega_T .$$

Expand the function $t \mapsto \det(I - Ad(t))_{\mathfrak{m}}$ in a sum of characters of T: $n_0\chi_0 + n_1\chi_1 + \cdots + n_k\chi_k$. Here χ_0 is the trivial character of T, appearing n_0 times, and for $i > 0$ each χ_i is a nontrivial character appearing n_i times. Taking for f the constant function equal to one, and applying the basic principle of invariants to T, we find $n_0 = |W|$.

Taking for f the function $f(g) = \det(I + Ad(g))$, i.e., the trace of $Ad(g)$ acting on $\Lambda\mathfrak{g}$, we find, using the Cartan-de Rham isomorphism (4.3), that

$$\dim H(G) = \dim(\Lambda\mathfrak{g})^G = \int_G \det(I + Ad(g)\omega_G$$

$$= \frac{1}{|W|}\int_T \det(I + Ad(t))\det(I - Ad(t))_{\mathfrak{m}}\,\omega_T$$

$$= \frac{2^{\dim T}}{|W|}\int_T \det(I - Ad(t^2))_{\mathfrak{m}}\,\omega_T\,.$$

Now the squaring map on T is surjective, so the square of a nontrivial character of T is still nontrivial. Hence the trivial character again appears with multiplicity $|W|$ in the expansion of $\det(I - Ad(t^2))_{\mathfrak{m}}$. This multiplicity is the value of the integral, so $\dim H(G) = 2^{\dim T} = 2^l$.

On the other hand, we saw in (5.3) that the trace of $w \in W$ acting on $H(G/T)$ is $|W|$ if $w = 1$, zero otherwise. Applying the invariance formula one more time, we find that $\dim[H(G/T) \otimes H(T)]^W = 2^l$ as well, completing the proof of (6.1). $\square$

We now have the main result

(6.2) THEOREM. *The cohomology ring* $H(G)$ *with real coefficients is a bigraded exterior algebra with generators in bi-degrees* $(2m_i, 1)$, *for* $1 \leqslant i \leqslant l$.

Proof. By (6.1) and (5.4), we have

$$H(G) \simeq [H(G/T) \otimes H(T)]^W \simeq [\mathscr{H}_{(2)} \otimes \Lambda]^W,$$

and by (3.8), the latter space is an exterior algebra with generators in degrees $(2m_i, 1)$, for $1 \leqslant i \leqslant l$. $\square$

Moreover, from the multiplicity formula (3.8), the dimensions of the bi-graded pieces are given in terms of the exponents as follows

(6.3) COROLLARY. *For each* $q \geqslant 0$, *we have*

$$\sum_{n=0}^{\dim G} \dim\,[H^{n-q}(G/T) \otimes H^q(T)]^W u^n = u^q s_q(u^{2m_1}, \ldots, u^{2m_l}) .$$

(6.4) We give two interpretations of the bigrading. First, we follow [L] and consider the spectral sequence of the fibration $G \to G/T$, which has E_2-term

$$E_2^{pq} = H^p(G/T) \otimes H^q(T) ,$$

and converges to $H(G)$. This spectral sequence does not degenerate at E_2, but it has a spectral subsequence which does degenerate, and still converges to $H(G)$.

To see this we again consider the Weyl group action. More precisely, N acts by automorphisms of the fibration $G \to G/T$, which in turn induce automorphisms of each term in the spectral sequence, commuting with the differentials. On $E_2^{pq} = H^p(G/T) \otimes H^q(T)$, the action of N factors through W and is the same as that considered above. Thus we have representations of W on the spaces E_2^{pq}, hence on each E_r^{pq} for $r \geqslant 2$.

For each p, q, r we decompose $E_r^{pq} = (E_r^{pq})^W \oplus (E_r^{pq})_W$, where the subscript W indicates a W-stable complement to the invariants. Each of the latter two spaces is a spectral subsequence, and since E_∞^{pq} is a subquotient of $H^{p+q}(G)$ and N acts trivially on $H(G)$ (because G is connected), we must have $(E_\infty^{pq})_W = 0$. On the other hand, $(E_\infty^{pq})^W$ is a subquotient of $(E_2^{pq})^W = [H^p(G/T) \otimes H^q(T)]^W$, so we have

$$\begin{aligned} 2^l = \dim H(G) &= \sum_{p,q} \dim\,(E_\infty^{pq})^W \leqslant \sum_{p,q} \dim\,(E_2^{pq})^W \\ &= \sum_q \dim\,[H(G/T) \otimes \Lambda^q]^W = 2^l . \end{aligned}$$

It follows that $\dim\,(E_\infty^{pq})^W = \dim\,(E_2^{pq})^W$ for all pq, so the spectral subsequence of W-invariants degenerates at $(E_2)^W$, and (6.1) is proved again.

(6.5) The significance of the bigrading on $H(G)$ can be seen in yet another way, inspired by [GHV]. We consider, for a fixed integer $k \neq 1$, the k^{th}-power maps $x \mapsto x^k$, denoted p_k and P_k, on T and G, respectively. It is shown in [GHV] that the Lefschetz number of P_k equals that of p_k, namely $(1-k)^l$. Let $H^n(G)_q$ be the k^q-eigenspace of P_k^* acting on $H^n(G)$. It is further shown in [GHV] that $\sum_n \dim H^n(G)_q = \binom{l}{q}$. We can refine this by computing each $\dim H^n(G)_q$ separately. Consider the commutative diagram

$$\begin{array}{ccc} H(G) & \overset{\psi^*}{\to} & [H(G/T) \otimes H(T)]^W . \\ p_k^* \downarrow & & \downarrow 1 \otimes p_k^* \\ H(G) & \overset{\psi^*}{\to} & [H(G/T) \otimes H(T)]^W . \end{array}$$

Since p_k^* acts by k^q on $H^q(T)$, (6.1) implies that $H^n(G)_q \simeq [H^{n-q}(G/T) \otimes H^q(T)]^W$, and (6.3) gives the dimension of the latter space.

(6.6) This last interpretation of the bigrading shows that it is natural in the following sense. Suppose $f: K \to G$ is a homomorphism between two compact connected Lie groups. Since f commutes with the power maps P_k on G and K, the cohomology map f^* sends $H^n(G)_q$ to $H^n(K)_q$. Suppose for example that K is a closed connected subgroup of G and f is the inclusion map. Choose, as we may, a maximal torus T of G such that $S := T \cap K$ is a maximal torus of K. The restriction map $H(G) \to H(K)$ becomes, via (6.1), the map $[H(G/T) \otimes H(T)]^W \to [H(K/S) \otimes H(S)]^{W_K}$ induced by restriction on each factor, where W_K is the Weyl group of S in K.

(6.7) We close with the homology interpretation of (6.1), which says the homology map ψ_* induced by ψ is surjective. It follows that the homology of G is spanned by the cycles $[\psi(\bar{X}_w, T_I)] = \{gtg^{-1}: gT \in \bar{X}_w, t \in T_I\}$. Here $w \in W$, X_w is the Schubert cell (see (5.2)) and $T_I = \prod_{i \in I} T_i$, where $T = T_1 \times \cdots \times T_l$, with each $T_i \simeq S^1$. Using the results in [BGG], one can explicitly write down the action of W on $H_*(G/T)$ in terms of the Schubert cell basis, and this leads, in principle, to the linear relations in $H_*(G)$ satisfied by the cycles $[\psi(\bar{X}_w, T_I)]$.

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Mark Reeder

University of Oklahoma
Dept. of Mathematics
Norman, Oklahoma 73019, USA
e-mail: mreeder@uoknor.edu