# 6. The Rigidity Theorem of Witten

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$w_\alpha^L(z)$ on $N_\alpha \cap L$ which represents the cohomology class $\widehat{\mathcal{A}}(N_\alpha \cap L)B(N_\alpha \cap L, z)$. Then $w_\alpha^L(z)$ is the form $a_\alpha^L$ given in the foliation Lefschetz theorem for $z$ acting on the leafwise Spin complex, and it defines a smooth form $w_\alpha(z)$ on $N_\alpha$. Thus for $z \in S^1$, $z$ not a root of unity, we have

$$L(z) = \int_N w(z) = \sum_\alpha \int_{N_\alpha} w_\alpha(z) .$$

Now notice that the right side of this equation defines a function $A(F, z)$ on the complex plane with values in the Haefliger forms of $F$. Also note that $A(F, z)$ has poles only at roots of unity and no pole at $z = \infty$, since $w_\alpha(z)$ has poles only at roots of unity and no pole at $z = \infty$. Because of the factor of $(z^d)^{1/2}$, $A(F, 0) = 0$. For $z \in S^1$, $z$ not a root of unity, $A(F, z) = L(z)$. But $L(z)$ is defined for all $z \in S^1$ and by Theorem 5.5 it is continuous on $S^1$. Thus $A(F, z)$ has no poles at all. Since it is analytic and bounded, it is constant and hence is identically zero. Therefore $L(z) = 0$ for all $z \in S^1$, but $L(1) = \widehat{\mathcal{A}}(F)$ so we are done.

The compactness of $G$ is essential, as in [HL 2], we give an example of an infinite discrete group acting by leaf preserving isometries on a compact oriented foliated manifold $M, F$ and $G$ preserves a Spin structure on $F$. The foliation $F$ admits an invariant transverse measure which defines a map from the Haefliger zero forms of $F$ to $\mathbf{C}$. The image of $\widehat{\mathcal{A}}(F)$ under this map is non-zero, so $\widehat{\mathcal{A}}(F) \neq 0$ .

## 6. The Rigidity Theorem of Witten

In 1986, Witten [W] predicted rigidity theorems for the indices of certain elliptic operators on manifolds with $S^1$ actions. The genesis for Witten's conjecture was his study of the Dirac operator on the free loop space $\mathcal{L}M$ (an infinite dimensional manifold) of a Spin manifold $M$. $\mathcal{L}M$ admits a natural $S^1$ action whose fixed point set is diffeomorphic to $M$. The sequences of bundles $R(q)$ and $R'(q)$ described below were derived from the normal bundle of $M$ in $\mathcal{L}M$ and from the formal analogue on $\mathcal{L}M$ of the fixed point formula for the Dirac operator in the finite dimensional case.

Let $D : C^\infty(E_1) \to C^\infty(E_2)$ be an elliptic operator on a compact manifold $M$ and suppose $M$ admits an $S^1$ action preserving $D$. Then as noted above, $\text{Index}(D)$ is a virtual $S^1$ module and has a decomposition into a finite sum of irreducible complex one dimensional representations

$$\mathrm{Index}(D) = \sum a_m L^m$$

where $z \in S^1$ acts on $L^m$ by multiplication by $z^m$. $D$ is called *rigid* if all the $a_m$ for $m \neq 0$ are zero, i.e. if the representation $L^m$, $m \neq 0$ occurs in kernel $D$ with multiplicity $a$ then it occurs in cokernel $D$ with the same multiplicity $a$.

Denote by $S^k(T)$ and $\lambda^k(T)$ the $k$th symmetric and exterior powers of $T = TM$ and set

$$S_a(T) = \sum_{k=0}^{\infty} a^k S^k(T)$$

$$\lambda_a(T) = \sum_{k=0}^{\infty} a^k \lambda^k(T).$$

Let $R_n$ and $R'_n$ be the sequences of bundles defined by the formal power series

$$R(q) = \sum_{n=0}^{\infty} q^n R_n = \bigotimes_{\ell=1}^{\infty} \lambda_{q^\ell}(T) \bigotimes_{m=1}^{\infty} S_{q^m}(T)$$

$$R'(q) = \sum_{n=0}^{\infty} q^{n/2} R'_n = \bigotimes_{\ell=\frac{1}{2},\frac{3}{2},\cdots} \lambda_{q^\ell}(T) \bigotimes_{m=1}^{\infty} S_{q^m}(T)$$

Now suppose $M$ is a $2n$ dimensional compact Riemannian Spin manifold and denote by $D^+$ the Dirac operator of $M$. For each $n$ we may form the operators

$$D^+ \otimes (E^+ \oplus E^-) \otimes R_n \quad \text{and} \quad D^+ \otimes R'_n.$$

THEOREM 6.1. *These operators are rigid under any $S^1$ action on $M$ by isometries, i.e. the induced action on the index of any of these operators is the trivial action.*

This is the theorem conjectured by Witten and first proven by Taubes [T]. A beautiful proof of it appears in [BT].

Roughly speaking Bott and Taubes' proof goes as follows. First they show that the Signature operator $d_S = D^+ \otimes (E^+ \oplus E^-)$ is rigid by an argument similar to that presented above. Combining this result with the power series $R(q)$ and interpreting ch(Index$\big(d_S \otimes R(q)\big)$ as a meromorphic function on the complex torus $T_{q^2} = \mathbf{C}^*/q^2$, they show that it has poles only at roots of unity and no poles on a certain circle $S^1 \subset T_{q^2}$. The Spin hypothesis then implies that it has no poles at all and hence is constant. Thus the character of $S^1$ given by its action on Index$(d_S \otimes R(q))$ is constant, and so the action must be trivial as claimed. They then give separate arguments to extend this result to $D^+ \otimes R'_n$.

These results all extend in a straight forward way to $S^1$ actions preserving a foliation (see [HL 2]).

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