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SYSTEMS OF CURVES ON A CLOSED ORIENTABLE SURFACE

by Allan L. EDMONDS

1. INTRODUCTION

It is well-known that a nontrivial one-dimensional homology class on a closed orientable surface F is represented by a simple closed curve in F if and only if it is *primitive*, i.e., indivisible. See Myerson [1976], Bennequin [1977], and Meeks-Patrusky [1978]. (There is also a partial result in Kaneko-Aoki-Kobayashi [1963].) Here we study the more general question of when a *collection* of pairwise distinct homology classes is represented by a set of corresponding pairwise disjoint simple closed curves. We first introduce the following necessary conditions.

THEOREM 1. *Let F be a closed orientable surface and let $S \subset H_1(F)$ be a set of pairwise distinct nonzero homology classes. If S is represented by a corresponding set of pairwise disjoint simple closed curves in F then the following three conditions are satisfied:*

1. INTERSECTION CONDITION. *The intersection pairing of F vanishes on S.*

2. SUMMAND CONDITION. *Every subset T of S spans a direct summand* span T *of $H_1(F)$.*

3. SIZE CONDITION. *For every subset T of S of more than one element* card $T \leq 3$ rank span $T - 3$.

Here we say that two homology classes α and β are *distinct* if $\alpha \neq \beta$ and $\alpha \neq -\beta$. Although linear algebraic in nature, the Summand Condition

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and Size Condition can be a little tricky to check in specific cases. None of these three conditions follows from the others in general.

We then investigate the attractive conjecture that these natural necessary conditions are in fact sufficient. In this direction we begin with the case of independent homology classes.

THEOREM 2. *Let F be a closed orientable surface and let $S \subset H_1(F)$ be a set of pairwise distinct, independent, and indivisible homology classes. Then S is represented by a corresponding set of pairwise disjoint simple closed curves in F if and only if the Intersection Condition holds and S spans a summand of $H_1(F)$.*

Here is a simple interpretation of the Theorem 2 in the case of just two homology classes, which is the case with which the present investigation started.

COROLLARY. *Let F be a closed orientable surface of genus $g \geq 2$ and let $\alpha_1, \alpha_2 \in H_1(F)$ be two distinct homology classes. Then α_1 and α_2 are represented by disjoint simple closed curves in F if and only if α_1 and α_2 are indivisible, $\alpha_1 \cdot \alpha_2 = 0$, and α_2 is indivisible in $H_1(F)/(\alpha_1)$.)*

Here are some simple interpretations of these basic results. Let

$$\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g$$

denote a standard symplectic basis for the homology of F. In particular, this means that these homology classes are represented by simple closed curves

$$A_1, B_1, A_2, B_2, \ldots, A_g, B_g$$

in F such that the A_i are pairwise disjoint, the B_j are pairwise disjoint, and if $A_i \cap B_j \neq \varnothing$, then $i = j$ and $A_i \cap B_j =$ a single point of transverse intersection. Then the corollary says that α_1 and $2\alpha_1 + \alpha_2$ are represented by disjoint simple closed curves, as one can easily check by hand, drawing suitable pictures. On the other hand, α_1 and $\alpha_1 + 2\alpha_2$ are not so represented. Note further that one can represent the three classes α_1, α_2, and $\alpha_1 + \alpha_2$ by disjoint simple closed curves, by explicitly drawing the curves. By Theorem 1, no more than 3 such classes can be so represented on a surface of genus 2. On a surface of genus 3 one can easily construct 6 pairwise disjoint simple closed curves representing distinct homology classes.

Again, Theorem 1 implies that on a surface of genus 3 one cannot realize 7 distinct classes this way. And so on. More subtle examples will be discussed later.

To consider the more general cases of not-necessarily independent homology classes we introduce the following terminology. Define the *rank* of S, rank S, to be the rank of the integral span of S in $H_1(F)$. And define the *excess* of S, excess S, to be card $S -$ rank S. Through a fairly painstaking and increasingly subtle analysis we are able to prove sufficiency of the conditions above when either the excess or rank is not too big.

THEOREM 3. *Let F be a closed orientable surface and let $S \subset H_1(F)$ be a set of pairwise distinct nonzero homology classes satisfying the Intersection Condition, Summand Condition, and Size Condition. Then S is represented by corresponding pairwise disjoint simple closed curves in F provided that either* excess $S \leq 3$ *or* rank $S \leq 4$.

The increasing difficulties encountered while attempting to extend the result of Theorem 3 eventually led to a family of counterexamples as described in the following result.

THEOREM 4. *Let F be a closed orientable surface of genus at least 5. Then there is a family $S \subset H_1(F)$ of 9 pairwise distinct nonzero homology classes satisfying the Intersection Condition, Summand Condition, and Size Condition and having excess 4 and rank 5 that is not representable by a corresponding family of pairwise disjoint simple closed curves in F.*

In particular Theorem 4 destroys all sorts of natural inductive approaches to proving realizability of families of homology classes by pairwise disjoint simple closed curves. We include in Section 7 of this paper some additional examples that illustrate the difficulties in proving realizability, including an example of realizable homology classes such that there is a realization of all but one of them that cannot be extended to a realization of the whole family.

A natural hope would be that perhaps the necessary conditions in Theorem 1 are at least sufficient after suitable stabilization or connected sum with a suitable number of tori. But this turns out not to be the case.

THEOREM 5. *Let F be a closed orientable surface and let $S \subset H_1(F)$ be a set of pairwise distinct nonzero homology classes satisfying the Intersection Condition, Summand Condition, and Size Condition. Suppose that the corresponding set of homology classes in $F \# F_1$ is represented by corresponding pairwise disjoint simple closed curves in $F \# F_1$ for some closed orientable surface F_1. Then S is represented by corresponding pairwise disjoint simple closed curves in F.*

The analysis in Theorems 3 and 4 is based upon realizing a maximal subcollection of independent classes by simple closed curves and then cutting open the given surface to form a surface with boundary. This surface can effectively be viewed as being planar. Then the problem of realizing any remaining classes is reduced to lifting the classes to homology classes in the punctured surface (which are not uniquely defined) and realizing them there. Thus we also include a preliminary step in which we give a complete analysis of the corresponding but much easier problem of realizing a family of homology classes in a compact planar surface by pairwise disjoint simple closed curves. One attractive statement in this context is that a family of homology classes in a planar surface is realizable by a corresponding family of pairwise disjoint simple closed curves if and only if each subcollection of two elements is so realizable. The analogue of this statement for closed surfaces is false.

An important consequence of this analysis of planar surfaces is the following result.

THEOREM 6. *Let F be a closed orientable surface and let $S \subset H_1(F)$ be a set of pairwise distinct nonzero homology classes satisfying the Intersection Condition, Summand Condition, and Size Condition. Then there is a finite (but "exponential") algorithm for deciding whether S can be represented by corresponding pairwise disjoint simple closed curves in F.*

We have written computer programs in Maple that in principle can carry out such an algorithm. Unfortunately, at the time of this writing the first interesting cases are too large for the current version of the programs to be effective. (The program did assist in enumerating the cases where rank $S = 4$ that were analyzed in Theorem 3.)

It is elementary to see that any family of homotopically nontrivial and nonparallel pairwise disjoint simple closed curves in a surface of genus g can be extended to a maximal family of $3g - 3$ such simple closed curves.

We conclude by proving an analogue of this for homologically nontrivial and distinct curves.

THEOREM 7. *Let F be a closed orientable surface of genus $g \geq 2$, and let $S \subset H_1(F)$ be a set of pairwise distinct nonzero homology classes represented by a corresponding family of pairwise disjoint simple closed curves in F. Then this family of simple closed curves can be extended to a family of $3g-3$ pairwise disjoint simple closed curves in F representing nontrivial, pairwise distinct homology classes in $H_1(F)$.*

Here is a summary of the contents of the rest of the paper: Section 2 contains the proof of Theorem 1 deriving the fundamental necessary conditions. Sections 3 and 4 deal with the cases of one homology class and with independent homology classes, and provide two proofs of Theorem 2. In Section 5 we give an analysis of simple closed curves on a planar surface. In Section 6 there is the proof of the main positive realizability statement, Theorem 3, ending with a discussion of Theorem 6. In Section 7 we present several examples that illustrate some of the subtleties involved in finding a more complete and definitive result than that given here, including the nonrealizability result stated as Theorem 4. Finally in Section 8 we give the proofs of Theorem 5 and 7.

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2. NECESSARY CONDITIONS

It is quite clear that the Intersection Condition must hold, since the intersection number of two disjoint 1-cycles is necessarily 0. The necessity of the Summand Condition follows immediately from the following lemma.

LEMMA 2.1. *Let F be a closed orientable surface of genus $g \geq 1$ and let $S \subset H_1(F)$ be a set of pairwise distinct homology classes represented by a corresponding set of pairwise disjoint simple closed curves in F. Then the image of S spans a direct summand of $H_1(F)$.*

Proof. Let $A \subset F$ be the union of the simple closed curves representing the elements of S in $H_1(F)$. Consider the long exact homology sequence of the pair (F, A).

$$\cdots \to H_1(A) \to H_1(F) \to H_1(F, A) \to \cdots$$

Now the linear span of S in $H_1(F)$ is identified with the image of $H_1(A)$ in $H_1(F)$. But $H_1(F, A)$ is free (by Poincaré Duality), so we see that the image of $H_1(A)$ is a direct summand, as required.

The following result gives the Size Condition. The construction described in the proof below will be important, as it describes an effective way to approach the problem of explicitly realizing a system of pairwise disjoint curves.

LEMMA 2.2. *Let F be a closed orientable surface of genus $g \geq 1$, let $S \subset H_1(F)$ be a set of pairwise distinct homology classes represented by a corresponding set of pairwise disjoint simple closed curves in F, and let $n =$ rank span S. Then* card $S \leq \max\{3n-3, 1\}$.

Proof. If $n = 1$, then it follows from Lemma 2.1 that S must consist of a single element, and the desired inequality trivially holds. Henceforth we assume that $n > 1$. The proof in this case will amount to cutting the surface up into pieces along the given simple closed curves and using the pieces to calculate the euler characteristic of the surface. It is easy to see that $g \geq n$. We will first assume that $g = n$. At the end we will indicate how to modify the proof to handle the case $g > n$.

Let $\alpha_1, \dots, \alpha_n \in S$ form a basis for span S and let $A_1, \dots, A_n$ be the corresponding disjoint simple closed curves in F. Let $\gamma_1, \dots, \gamma_m \in S$ be the remaining elements of S and $C_1, \dots, C_m$ be the corresponding disjoint simple closed curves in F. Let $\widehat{F}$ denote the surface F cut open along the A_i. Then $\widehat{F}$ is a connected, orientable surface and has $2n$ boundary curves and genus $g - n = 0$. Note that $\chi(\widehat{F}) = \chi(F)$ by the sum formula for euler characteristics. In $\widehat{F}$ each of the $m =$ card $S - n$ curves C_j is homologous to a sum of boundary curves, with multiplicities ± 1, since C_j does not separate F, but does separate $\widehat{F}$. Now $\overline{F} = \widehat{F} - \cup C_j$ still has genus 0 and consists of $m+1$ planar components X_ℓ. Again note that $\chi(\overline{F}) = \chi(\widehat{F})$. No X_ℓ can be a disk, since otherwise its boundary curve would be nullhomologous in F. Similarly, no X_ℓ can be an annulus, since otherwise, the two boundary curves, belonging to the original collection of curves would represent the same homology class in F, up to sign. It follows, therefore, from the classification of surfaces, that each X_ℓ has Euler characteristic ≤ -1. Therefore, when $n > 1$ and $n = g$,

$$\chi(F) = \chi(\bar{F}) = 2 - 2n = \sum \chi(X_\ell) \leq (\text{card } S - n + 1)(-1)$$

or equivalently card $S \leq 3n - 3$, as required.

It remains to consider the case when $g > n > 1$. In this case, we first proceed as before, cutting open along the A_i, obtaining a connected surface $\widehat{F}$ of genus $g-n$ and with $2n$ boundary curves, containing the m curves C_j, each of which is homologous to a sum of boundary curves in $\widehat{F}$. Now each of the C_j separates $\widehat{F}$, and we may further cut open along the C_j, obtaining a surface with $m+1$ components and total genus $g-n$. It follows that there are additional pairwise disjoint simple closed curves E_k, $k = 1, \ldots, g-n$, in $\widehat{F}$, reducing $\widehat{F}$ to a planar surface of genus 0 when we cut open along the E_k and cap off the resulting $2(g-n)$ boundary curves with disks. Call this latter surface $\bar{F}$, topologically a 2-sphere with $2n$ holes. Now the C_j separate $\bar{F}$ into $m + 1 =$ card $S - n + 1$ planar components X_ℓ. As before, each X_ℓ has Euler characteristic ≤ -1. Therefore, again,

$$\chi(\bar{F}) = 2 - 2n = \sum \chi(X_\ell) \leq (\text{card } S - n + 1)(-1)$$

or equivalently card $S \leq 3n - 3$, as required.

3. Sufficiency for a single homology class

Here we collect some basic information about the embedding of a single simple closed curve in a surface, and offer an alternative, elementary proof of Theorem 2 for the well-known case of a single homology class.

LEMMA 3.1. *A nonzero homology class $\alpha \in H_1(F)$ is primitive if and only if there exists $\gamma \in H_1(F)$ such that $\gamma \cdot \alpha = 1$.*

Proof. A nonzero element of a finitely generated free abelian group is primitive if and only if it is part of a basis if and only if there is a $\mathbf{Z}$-valued homomorphism that takes the value 1 on it. Recall that taking intersection numbers of 1-cycles defines a skew symmetric bilinear form on $H_1(F)$. The content of Poincaré Duality in this situation is that this bilinear form is nonsingular, that is, the adjoint homomorphism $H_1(F) \to \operatorname{Hom}\big(H_1(F), \mathbf{Z}\big)$ is an isomorphism. The lemma then follows.

LEMMA 3.2. *Any homology class $\alpha \in H_1(F)$ can be represented by an immersed, oriented closed curve on F and also by an embedded, oriented 1-submanifold.*

Proof. The Hurewicz homomorphism $\pi_1(F) \to H_1(F)$ is onto. Compare W. Massey [1980], Chapter III, Section 7, for example. Any map $S^1 \to F$ can be approximated by an immersion, with only isolated double points. One can surger any double points, that is, one can replace any pair of small oriented arcs having a single transverse intersection with a pair of parallel oriented arcs with the same end points and lying within a regular neighborhood of the intersecting arcs. In this way one creates a disjoint union of oriented simple closed curves representing the same homology class.

PROPOSITION 3.3. *A homology class α in $H_1(F)$ can be represented by a simple closed curve on F if and only if α is primitive.*

Proof. We sketch a 2-dimensional version of the argument of Bennequin [1977]. If a simple closed curve represents a nonzero homology class, then it is nonseparating. It follows that there is a simple closed curve that meets it transversely in a single point. This implies indivisibility, by the homology invariance of intersection numbers.

For the converse, we may assume that α is nonzero. We begin by representing α by a disjoint union A of oriented simple closed curves, as in Lemma 3.2. We shall assume that A contains the smallest possible number of components and show that this number can be reduced unless it is 1 or it is equal to the divisibility of α.

Cut open F along A–that is, remove the interior of a small tubular neighborhood of A. The boundary of the cut open surface $\widehat{F}$ consists of two copies A_i^+ and A_i^- of each component A_i of A, each of which we orient as the boundary of the orientable surface $\widehat{F}$. The positive components A_i^+ have the same orientation as A_i, while the negative components A_i^- have the opposite orientation.

If some component R of $\widehat{F}$ contains in its boundary two positive curves A_i^+ and A_j^+ (or two negative curves), then they can be banded together in an orientable way using a band in R. That is, one chooses an embedded arc δ in R meeting A_i^+ and A_j^+ in its two end points only. One then replaces A_i and A_j with the single simple closed curve obtained by removing small arcs in A_i and A_j containing the end points of δ and inserting in their place two embedded arcs parallel to δ. This would reduce the number of components of A. If some component R has boundary just A_i^+ and A_i^- for some A_i, then we can conclude that A is connected and we are done. If some R has more than two boundary components, then it contains two positive curves or two negative curves, and we can proceed as above to reduce the number of components of A.

It remains to consider the case where each component R_k of $\widehat{F}$ has exactly two boundary components of the form A_i^+ and A_j^-, where A_i and A_j are distinct components of A. In this case we conclude that we can arrange the components of A in a sequence $A_1, A_2, \ldots, A_n$, so that A_1 is homologous to A_2, A_2 is homologous to $A_3, \ldots, A_n$ is homologous to A_1. In this case, then, the number n of components is exactly the divisibility of α.

4. Sufficiency for independent homology classes

In this section we complete the proof of Theorem 2, dealing with the case of a set of homology classes consisting of independent elements.

LEMMA 4.1. *Let F be a closed orientable surface and let $\alpha_1, \ldots, \alpha_n \in H_1(F)$ be independent homology classes that span a summand of $H_1(F)$ on which the intersection pairing of F vanishes. Then there exists $\gamma \in H_1(F)$ such that $\gamma \cdot \alpha_n = 1$ and $\gamma \cdot \alpha_i = 0$ for $i < n$.*

Proof. This is a consequence of Poincaré Duality.

PROPOSITION 4.2. *Let F be a closed orientable surface and let $\alpha_1, \ldots, \alpha_n \in H_1(F)$ be independent homology classes that span a summand of $H_1(F)$ on which the intersection pairing of F vanishes. Then there exist pairwise disjoint simple closed curves $A_1, \ldots, A_n$ in F representing the homology classes $\alpha_1, \ldots, \alpha_n$.*

Proof. The proof will proceed by induction on n. The case $n = 1$ is given by Proposition 3.3.

Now inductively consider the case of $n > 1$ homology classes. By Proposition 3.3 we can find a simple closed curve A_n in F representing α_n. We claim that there is a simple closed curve B_n in F representing a homology class β_n such that B_n meets A_n in exactly one point and such that $[B_n] \cdot \alpha_i = 0$ for $i < n$. By Lemma 4.1 there is a homology class $\gamma_n \in H_1(F)$ such that $\alpha_i \cdot \gamma_n = \delta_{i,n}$. We begin by representing γ_n by a simple closed curve B transverse to A_n. By tubing together neighboring pairs of intersection of B with A_n of opposite sign we can transform B into a disjoint union B'

of simple closed curves meeting A_n in exactly one point. Now we can band together the components of B', using bands in the complement of A_n to create a closed curve B'' representing γ_n and meeting A_n in exactly one point. But B'' may now have self-intersections. We may then eliminate the self-intersections by sliding segments of B'' over A_n. This creates a simple closed curve B_n meeting A_n in exactly one point, and representing a homology class of the form $\beta_n = \gamma_n + k\alpha_n$, which proves the claim.

Now the union of the two curves A_n and B_n has a small neighborhood N of the form of a once punctured torus. Let F_n denote the result of removing N and replacing it with a disk D. Then $F_n - D = F - N \subset F$ and inclusion identifies $H_1(F_n)$ with the orthogonal complement of α_n and β_n in $H_1(F)$. Thus the homology classes $\alpha_1, \ldots, \alpha_{n-1}$ determine well-defined classes in $H_1(F_n)$, which we continue to call by the same names. By induction there are pairwise disjoint simple closed curves $A_1, \ldots, A_{n-1}$ in F_n representing the homology classes $\alpha_1, \ldots, \alpha_{n-1}$. Then these curves also live in F, determining the same homology classes, and are disjoint from the curve A_n. This completes the proof.

Here is a sketch of a standard but somewhat more learned proof of Proposition 4.2, suggested by M. Kervaire: Any basis for a self-annihilating summand of a skew-symmetric inner product space over $\mathbf{Z}$ can be extended to be part of a symplectic basis. Any two symplectic bases are related by an isometry of the inner product space. Half of a fixed standard symplectic basis is clearly represented by standard pairwise disjoint simple closed curves in a standard model of the surface. And any isometry is induced by a homeomorphism of the surface, so that the standard curves are taken to the desired curves. To see that any isometry is induced by a homeomorphism one can argue that the symplectic group is generated by certain elementary automorphisms and that these elementary automorphisms are induced by Dehn twist homeomorphisms around standard curves on the surface.

5. Disjoint simple closed curves on a planar surface

Subsequent proofs of realizability of non-independent homology classes will proceed by cutting the surface along curves representing a basis for homology until it becomes a punctured 2-sphere and then representing the remaining homology classes by disjoint curves on this planar surface. We therefore first investigate directly the case of homology classes in a planar surface.

In this section G will denote a compact orientable planar surface with with m oriented boundary components $B_1, \ldots, B_m$. By the classification of surfaces G can be thought of as being obtained from the 2-sphere by removing the interiors of m disjoint disks. Now $H_1(G)$ is freely generated by the homology classes $[B_i]$, subject to the single relation $\sum_i [B_i] = 0$.

By the Schönflies Theorem any simply closed curve C in G divides G into 2 parts, showing that any such C is homologous to a sum of boundary curves, up to global sign. That is, we have half of the following lemma.

LEMMA 5.1. *A homology class $\gamma = \sum \varepsilon_i [B_i] \in H_1(G)$ is represented by a simple closed curve if and only if $\varepsilon_i \in \{0, +1\}$ for all i, or $\varepsilon_i \in \{0, -1\}$ for all i.*

Proof. It remains to show that a given $\gamma = \sum \varepsilon_i [B_i] \in H_1(G)$, with $\varepsilon_i \in \{0, 1\}$ is represented by a simple closed curve. One can organize this process by choosing a tree in G meeting only the boundary curves B_i with coefficient $\varepsilon_i = 1$, and then only in one point for each such B_i. The desired simple closed curve can be chosen to be the interior boundary of a small regular neighborhood of the union of the tree and the boundary curves it meets.

Note, for example, that $[B_1] + [B_2]$ is represented by a simple closed curve, while $[B_1] - [B_2]$ and $[B_1] + 2[B_2]$ are not.

We call a homology class, as in the statement of Lemma 5.1 a *basic* class. Notice that if γ is basic, then so is $-\gamma$. Notice also that a nonzero basic class has a unique representation with all nonnegative coefficients. There are $2^m - 1$ nonzero basic classes, then, that we want to consider.

We now consider a family of homology classes $\gamma_1, \ldots, \gamma_k \in H_1(G)$ and ask when they can be represented by pairwise disjoint simple closed curves in G. Using the above lemma together with the fundamental defining relation for the homology of G we may as well assume that each

$$\gamma_i = \sum \varepsilon_{ij} [B_j]$$

where each $\varepsilon_{ij} \in \{0, 1\}$. Now all intersection numbers in G necessarily vanish, so there is no analogue of the Intersection Condition from Theorem 1.

If $\alpha = \sum \varepsilon_i [B_i] \in H_1(G)$ then we define the *partition* of α, denoted part $\alpha = \{C, D\}$, to be the partition of the set $\{[B_i] : i = 1, \ldots, n\}$ consisting of the set C of $[B_i]$ that have nonzero coefficients ε_i and its complement D. Note that since the representation of α as such a linear combination is not unique, it is necessary to include discussion of the complementary sum. Note also that α and $-\alpha$ have the same partitions.

PROPOSITION 5.2. *Two basic classes $\alpha_1, \alpha_2 \in H_1(G)$, with corresponding partitions* part $\alpha_i = \{C_i, D_i\}$*, are represented by two disjoint simple closed curves in G if and only if α_1 and α_2 are individually represented by simple closed curves and $C_1 \subset C_2$ or $C_1 \subset D_2$.*

Proof sketch. The point is that the tree used to determine a simple closed curve A_1 for α_1 does not separate G. Therefore, if the support of α_2, or its complement, is disjoint from the support of α_1, then one can find a tree for α_2 in the complement of the tree for α_1 and the boundary curves it touches.

The proof of Proposition 5.2, extends inductively to prove the following result.

PROPOSITION 5.3. *A set of homology classes $S = \{\alpha_1, \ldots, \alpha_k\} \subset H_1(G)$, with corresponding partitions* part$(\alpha_i) = \{C_i, D_i\}$*, is represented by a corresponding set of pairwise disjoint simple closed curves in G if and only if each α_i is individually represented by a simple closed curve and for each i,j $C_i \subset C_j$ or $C_i \subset D_j$.*

COROLLARY 5.4. *A set S of homology classes in $H_1(G)$ is represented by pairwise disjoint simple closed curves in G if and only if any two elements of S are represented by disjoint simple closed curves in G.*

The analogue of the preceding result will be seen to fail for closed surfaces.

COROLLARY 5.5. *A set S of pairwise distinct homology classes in $H_1(G)$ that is represented by pairwise disjoint simple closed curves in G has at most* $2m - 3$ *elements, including the boundary curves.*

Proof. It suffices to assume that S contains no classes homologous to boundary curves and to show that card $S \leq m - 3$. Let $k =$ card S and let A denote the union of a set of disjoint simple closed curves in G representing the elements of S. The realization of each element of S divides G into 2 parts. The k elements then divide G into $k+1$ parts X_ℓ. Since the classes in S are not parallel to boundary classes, the components X_ℓ of G cut open along the simple closed curves representing the elements of S all have negative Euler characteristic. Therefore $2-m=\chi(G)=\sum_\ell \chi(X_\ell)\leq(k+1)(-1)$. It follows that $k\leq m-3$, as required.

In general there are many apparently different ways to realize realizable homology classes. But up to homeomorphism we have the following uniqueness result.

THEOREM 5.6. *Suppose that a set of homology classes $S=\{\alpha_1,\dots,\alpha_k\}\subset H_1(G)$ is represented by two different families $A_1,A_2,\dots,A_k$ and $A'_1,A'_2,\dots,A'_k$ of pairwise disjoint simple closed curves in the planar surface G. Then there is a homeomorphism $f:G\to G$ inducing the identity on homology such that $f(A_j)=A'_j$ for $j=1,\dots,k$.*

We note that the analogue of Theorem 5.6 for closed surfaces is false. We also note that this result shows that in the process of realizing a realizable family of homology classes one-by-one, one cannot get "stuck", which can in fact happen in the case of closed surfaces.

Proof of Theorem 5.6. The overall argument will be by induction on the the number k of homology classes in question. We can assume that G has at least 3 boundary curves. Then any homeomorphism inducing the identity on homology will map each boundary component into itself. It follows that we can assume that the set S of homology classes contains no boundary classes. First consider the case $k=1$ of just one non-boundary class α_1 and two different simple closed curves A_1 and A'_1 realizing it. Note that the same boundary curves appear on corresponding sides of A_1 and of A'_1. It follows easily from the Schönflies Theorem that there is a homeomorphism moving A_1 onto A'_1 and preserving the corresponding sides. One can then arrange that this homeomorphism induce the identity on the boundary by composing with a homeomorphism that appropriately permutes the boundary curves while leaving A'_1 invariant. To argue this in a little more detail, let $\overline{G}$ denote the 2-sphere obtained by collapsing all the boundary curves to single points. The the usual Schönflies Theorem shows that there is a homeomorphism of $\overline{G}$ mapping A_1 onto A'_1. By composing with a homeomorphism that exchanges the two sides of A'_1 if necessary, we can assume that this homeomorphism maps the complementary domains of A_1 to the corresponding complementary domains of A'_1. Then homogeneity shows that one can further arrange that this homeomorphism can be arranged to map each ideal point to itself. One can then "blow up" the ideal points to the original boundary curves.

Now, proceeding inductively, consider the case of $k > 1$ homology classes. One of these homology classes, say α_k, has a minimal partition C_k, D_k, in the sense that C_k contains no other C_j or D_j for j< k. By the preceding argument we may assume that $A_k = A'_k$. One side of A_k contains no other simple closed curves A_j or A'_j. Excise this side to obtain a new planar surface H containing the remaining simple closed curves. By induction there is a homeomorphism h of H moving A_j onto A'_j for $1 \leq j \leq k-1$. and mapping each boundary curve into itself. We can then reinsert the excised domain to complete the argument.

The results of this section, with the exception of Theorem 5.6 above, hold *mutatis mutandi* for compact non-planar surfaces G with boundary, provided one only considers homology classes given as linear combinations of the classes represented by the boundary curves. Each such simple closed curve in the interior of G would separate G. Uniqueness, however, is obstructed by needing to know the genus of each complementary domain.

6. Sufficiency in Theorem 3

Let $S \subset H_1(F)$ denote a finite set of distinct homology classes satisfying the Intersection Condition, the Summand Condition, and the Size Condition of Theorem 1, which we wish to represent by pairwise disjoint simple closed curves. Suppose that the linear span of S has rank n and extract from S n elements $\alpha_1, \ldots, \alpha_n$ that form a basis for this span. Now each element γ_i in the remaining subset T of S can be expressed as a linear combination

$$\gamma_i = \sum_j \varepsilon_{ij} \, \alpha_j \, .$$

We refer to the γ_i as "composite classes."

LEMMA 6.1. *Each coefficient ε_{ij} in the linear combination $\gamma_i = \sum_j \varepsilon_{ij} \, \alpha_j$ is 1, -1, or 0.*

Proof. To see this, consider the span of the set consisting of any one γ_i together with all α_k, $k \neq j$. Elementary change of basis operations show that this span is the same as the span of $\varepsilon_{ij} \, \alpha_j$ and the α_k, $k \neq j$. By the Summand Condition, this span must be a summand, and it therefore follows that $\varepsilon_{ij} = \pm 1$ or 0.

If card $T = m$, then the collection of all $\gamma_i \in T$ can be described by an m by n matrix M of 0's, 1's, and -1's. Then the proof of Lemma 6.1 extends to give the following consequence of the Summand Condition.

LEMMA 6.2. *Each square submatrix N of M has* $|\det N| \leq 1$.

Proof. Let N be a $k \times k$ submatrix. Up to relabeling we may assume that N consists of the ε_{ij}, $1 \leq i,j \leq k$. Now consider the span of $\gamma_1, \dots, \gamma_k, \alpha_{k+1}, \dots, \alpha_n$. On the one hand, the Summand Condition says that this span must be a direct summand. On the other hand, the span is the same as the span of $\tilde{\gamma}_1, \dots, \tilde{\gamma}_k, \alpha_{k+1}, \dots, \alpha_n$, where $\tilde{\gamma}_i = \sum_{j \leq k} \varepsilon_{ij} \alpha_j$ is the projection of γ_i into the span of $\alpha_1, \dots, \alpha_k$. But this span is clearly the direct sum of the span of $\tilde{\gamma}_1, \dots, \tilde{\gamma}_k$ and of $\alpha_{k+1}, \dots, \alpha_n$. It follows that the span of $\tilde{\gamma}_1, \dots, \tilde{\gamma}_k$ is a direct summand of the span of $\alpha_1, \dots, \alpha_k$. Standard matrix theory then implies that the determinant of the matrix of coefficients of $\tilde{\gamma}_1, \dots, \tilde{\gamma}_k$ is ± 1 or 0. But this matrix is the upper left matrix N.

In what follows here we will assume that F has genus n. By this we mean that there is a corresponding set of homology classes in a surface of genus n, and that we may view the given surface as being obtained from the genus n surface by attaching handles. It is clear that if the homology classes can be realized in the surface of genus n, then they can be realized in the given surface. The converse of this statement is also true, but less obvious. We will prove it in a subsequent section.

We may also assume that we have already represented elements $\alpha_1, \dots, \alpha_n$ by disjoint simple closed curves elements $A_1, \dots, A_n$, by Proposition 4.2. We attempt to represent the remaining classes in T, the complement of $\alpha_1, \dots, \alpha_n$ in S. Let $\widehat{F}$ denote F cut open along the A_i. Then $\widehat{F}$ is a 2-sphere with $2n$ holes, with orientable boundary consisting of one copy A_i^+ of each A_i and one copy A_i^- of each A_i with its orientation reversed.

By the results in Section 5 we understand completely when a family of homology classes in $\widehat{F}$ can be realized by pairwise disjoint simple closed curves. We need to see how to lift the classes in T to realizable class in $\widehat{F}$. Now $H_1(\widehat{F})$ is generated by the classes $[A_i^+]$ and $[A_i^-]$ subject to the single relation $\sum_i([A_i^+] + [A_i^-]) = 0$. The natural inclusion of $\widehat{F}$ in F induces a homomorphism $H_1(\widehat{F}) \to H_1(F)$ where $[A_i^+] \to [A_i]$ and $[A_i^-] \to -[A_i]$. This homomorphism maps surjectively onto the subgroup generated by $A_1, \dots, A_n$. Its kernel is generated by terms of the form $[A_i^+] + [A_i^-]$ subject to the single global relation $\sum_i([A_i^+]+[A_i^-]) = 0$. We will slightly abuse notation and suppress the square brackets from such formulas below.

LEMMA 6.3. *Any single element $\gamma_1 \in T$ can be realized by a simple closed curve in $\widehat{F}$.*

Proof. Write $\gamma_1 = \sum_j \varepsilon_{1j}\alpha_j$ as above, with $\varepsilon_{1j} \in \{0, \pm 1\}$. By replacing some of the α_j with $-\alpha_j$ as necessary, we can assume that $\gamma_1 = \sum_{j=1}^k \alpha_j$. The corresponding homology class $\widehat{\gamma_1} = \sum_{j=1}^k A_j^+$ in $\widehat{F}$ is then represented by a simple closed curve, as required.

LEMMA 6.4. *If α_j and α_k both have nonzero coefficients in the expansions of both of γ_1 and γ_2, then either $\varepsilon_{1j} = \varepsilon_{2j}$ and $\varepsilon_{1k} = \varepsilon_{2k}$ or $\varepsilon_{1j} = -\varepsilon_{2vj}$ and $\varepsilon_{1k} = -\varepsilon_{2k}$. That is, the coefficients either agree or disagree.*

Proof. If not, the matrix M representing the γ_i has a 2 by 2 submatrix of the form

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

up to multiplying rows and/or columns by -1, contradicting the matrix interpretation of the Summand Condition as given in Lemma 6.2.

For $\gamma_i \in T$ define its *support* (with respect to $\{\alpha_1, \dots, \alpha_n\}$) to be the set of α_j with nonzero coefficient in the expression $\gamma_i = \sum_j \varepsilon_{ij}\alpha_j$. Note that up to relabeling there are just three ways for the supports of γ_1 and γ_2 in T to relate to one another. The two classes may have nested supports, disjoint supports, or properly overlapping supports.

LEMMA 6.5. *Any two distinct elements $\gamma_1, \gamma_2 \in T$ can be realized by disjoint simple closed curves in $\widehat{F}$.*

Proof. There are three cases to consider, organized by the relative placement of the supports. Without loss of generality we can assume that card supp $\gamma_1 \geq$ card supp γ_2. Then either

(1) supp $\gamma_2 \subset$ supp γ_1 or

(2) supp $\gamma_1 \cap$ supp $\gamma_2 = \varnothing$ or

(3) supp $\gamma_1 \cap$ supp $\gamma_2 \neq \varnothing$ and supp$\gamma_1 \cap$ supp $\gamma_2 \neq$ supp γ_2.

As in the proof of Lemma 6.3 we may assume that $\gamma_1 = \sum_{j=1}^k \alpha_j$.

In case (1) we may, by Lemma 6.4, assume that γ_2 has the form $\sum_{j=1}^{\ell} \alpha_j$ for some $\ell < k$. Then the two corresponding classes $\widehat{\gamma}_1 = \sum_{j=1}^{k} A_j^+$ and $\widehat{\gamma}_2 = \sum_{j=1}^{\ell} A_j^+$ can be realized disjointly in $\widehat{F}$ as required, by Proposition 5.2.

In case (2) we may assume that γ_2 has the form $\sum_{j=k+1}^{\ell} \alpha_j$ for some $\ell > k$. Then the two corresponding classes $\widehat{\gamma}_1 = \sum_{j=1}^{k} A_j^+$ and $\widehat{\gamma}_2 = \sum_{j=k+1}^{\ell} A_j^+$ can be realized disjointly in $\widehat{F}$ as required.

In case (3) we may assume, again by Lemma 6.4, that γ_2 has the form $\sum_{j=r}^{s} \alpha_j$ for some $r \leq k$ and $s > k$. Then the two corresponding classes $\widehat{\gamma}_1 = \sum_{j=1}^{k} A_j^+$ and $\widehat{\gamma}_2 = \sum_{j=r}^{s} A_j^-$ can be realized disjointly in $\widehat{F}$ as required.

PROPOSITION 6.6. *Any three distinct elements γ_1, γ_2, γ_3 in T can be realized by disjoint simple closed curves in $\widehat{F}$.*

Proof. Once again we organize the analysis according to the relative positions of the supports of the three homology classes. There are several cases to consider. In each of several cases we shall normalize the expressions for the γ_i in terms of the α_j. The operations we will use are permutation of the γ_i, permutation of the α_j, changing the sign of one or more γ_i, and changing the sign of one or more α_j.

Suppose that the support of one class is contained in the support of another class. Without loss of generality we may assume that

$$\gamma_1 = \sum_{j=1}^{k} \alpha_j \quad \text{and} \quad \gamma_2 = \sum_{j=1}^{\ell} \alpha_j \quad \text{for some} \quad \ell < k.$$

Now the signs of all coefficients of γ_3 involved in γ_1 may be assumed to be negative, by Lemma 6.4. So we may assume that

$$\gamma_3 = \sum_{j=u}^{v} -\alpha_j$$

Then the three preferred lifts

$$\widehat{\gamma}_1 = \sum_{j=1}^{k} A_j^+ , \quad \widehat{\gamma}_2 = \sum_{j=1}^{\ell} A_j^+ , \quad \text{and} \quad \widehat{\gamma}_3 = \sum_{j=u}^{v} A_j^-$$

can clearly be realized disjointly in $\widehat{F}$. Henceforth we may assume that no one of the three given homology classes has its support contained in the support of one of the others.

If the underlying support of one of the 3 classes, say of γ_3, is disjoint from the supports of both of the other two, then this is easy. Realize $\widehat{\gamma_1}$ and $\widehat{\gamma_2}$ as above; then realize the preferred lift $\widehat{\gamma_3}$ of γ_3, which has support disjoint from those of $\widehat{\gamma_1}$ and $\widehat{\gamma_2}$.

Suppose now that two classes have disjoint support, but that no homology class has support disjoint from the supports of both of the other two. Without loss of generality we may assume that

$$\gamma_1 = \sum_{j=1}^{k} \alpha_j \quad \text{and} \quad \gamma_2 = \sum_{j=k+1}^{\ell} \alpha_j$$

for some $\ell > k+1$. Now γ_3 involves some, but not all, of the support of γ_1, some, but not all, of the support of γ_2, and, perhaps, some terms not involved in either of γ_1 or γ_2. After permuting basis elements we have $\gamma_3 = \sum_{j=r}^{s} \varepsilon_{3j}\alpha_j + \sum_{j=\ell+1}^{n} \varepsilon_{3j}\alpha_j$, where $2 \le r \le k-1$, $k+1 \le s \le \ell - 1$. Now the Summand Condition implies that all ε_{3j}, $r \le j \le k$, have the same sign; and all ε_{3j}, $k+1 \le j \le s$, have the same sign. By changing the global sign of γ_3 if necessary we may assume that $\varepsilon_{3j} = -1$ for $r \le j \le k$. Further, by changing the sign of α_i, $i > \ell$ as needed we may assume that $\varepsilon_{3j} \le 0$ for $i > \ell$. Thus at this point we have arranged that

$$\gamma_3 = -\sum_{j=r}^{k} \alpha_j \pm \sum_{j=k+1}^{s} \alpha_j - \sum_{j=\ell+1}^{t} \alpha_j$$

where $2 \le r \le k-1$, $k+1 \le s \le \ell - 1$, and $\ell + 1 \le t \le n$, and the third sum might not actually appear at all. If the "$-$" sign prevails in the formula for γ_3, then the preferred lifts of γ_1, γ_2, and γ_3 are disjointly realizable in $\widehat{F}$ as required. On the other hand, if the "$+$" sign prevails we can reduce to the previous case by first replacing $\alpha_{k+1}, \dots, \alpha_\ell$ with their negatives, and then replacing γ_2 with its negative.

Now we may suppose for the rest of the argument that no two classes have disjoint support, and that no class has support contained in the support of one of the other two classes.

For the penultimate case suppose that the intersection of all three supports is empty. We divide the supports of the γ_i three pieces : $S_{ij} = \operatorname{supp} \gamma_i \cap \operatorname{supp} \gamma_i$ and $T_i = \operatorname{supp} \gamma_i - \operatorname{supp} \gamma_j \cup \operatorname{supp} \gamma_k$, where $\{i, j, k\} = \{1, 2, 3\}$. In what follows we will, for notational simplicity, sometimes identify α_j with its index j. Then, without loss of generality, after changing the signs of various α_i as necessary, we can assume that

$$\gamma_1 = \sum_{i \in S_{12}} \alpha_i + \sum_{i \in S_{13}} \alpha_i + \sum_{i \in T_1} \alpha_i .$$

Then, replacing γ_2 by its negative if necessary, and changing the sign of α_i, $i \in S_{23} \cup T_2$ as necessary, and invoking the 2×2 Summand Condition, we can assume that

$$\gamma_2 = -\sum_{i \in S_{12}} \alpha_i - \sum_{i \in S_{23}} \alpha_i - \sum_{i \in T_2} \alpha_i .$$

Similarly, we can arrange that

$$\gamma_3 = -\sum_{i \in S_{13}} \alpha_i \pm \sum_{i \in S_{23}} \alpha_i + \sum_{i \in T_3} \alpha_i .$$

Now the 3×3 Summand Condition tells us that the $+$ sign must prevail in the expansion of γ_3. For otherwise the matrix M would contain a 3×3 submatrix of the form

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & -1 \end{pmatrix}$$

which has determinant -2. Now with all these normalizations, the preferred lifts of γ_1, γ_2, and γ_3 are disjointly realizable in $\widehat{F}$ as required.

Finally, at last, we have the case that the intersection of all three supports is nonempty but that no support set is contained in one of the other supports. Let $S_i = \operatorname{supp} \gamma_i$, $S_{ij} = S_i \cap S_j$, and $S_{123} = S_1 \cap S_2 \cap S_3 \neq \varnothing$. Now as always we can assume one of our homology classes, say γ_1 has all nonnegative coefficients, i.e., $\gamma_1 = \sum_{j \in S_1} \alpha_j$. Next we can assume by the 2 by 2 Summand Condition that γ_2 has positive coefficients on S_{12}, and of course that it has positive coefficients on $S_2 - S_{12}$. In particular, then, we have $\gamma_2 = \sum_{j \in S_2} \alpha_j$. Since $S_{123} \neq \varnothing$, all coefficients of elements of $S_3 \cap (S_1 \cup S_2)$ must have the same sign, which we may assume is positive. It follows that we may arrange that $\gamma_3 = \sum_{j \in S_3} \alpha_j$. In this case the preferred lifts of the γ_i will not be disjointly realizable and we have to choose other lifts as follows. For γ_1 we do use the preferred lift

$$\widehat{\gamma}_1 = \sum_{j \in S_1} A_j^+ .$$

For γ_2, however, we add on to the preferred lift canceling pairs corresponding to elements of $S_1 - S_2$ and define

$$\widehat{\gamma}_2 = \sum_{j \in S_2} A_j^+ + \sum_{j \in S_1 - S_2} (A_j^+ + A_j^-)$$

and, finally, for γ_3 we define

$$\widehat{\gamma}_3 = \sum_{j \in S_3} A_j^+ + \sum_{j \in S_1 \cup S_2 - S_3} (A_j^+ + A_j^-) .$$

These choices of lifts of the γ_i to homology classes in $\widehat{F}$ satisfy the conditions for disjoint realizability given in Section 5. (We emphasize again that the conditions for realizability in planar surfaces continue to hold for homology classes in nonplanar compact surfaces with boundary provided the homology classes in question are all linear combinations of the classes of the boundary curves.)

The one remaining aspect to consider in the proof of the sufficiency part of the Theorem 3 is given by the following result.

PROPOSITION 6.7. *If* rank $S \leq 4$, *then* S *can be realized by disjoint simple closed curves in* $\widehat{F}$.

Proof sketch. We will only outline the proof, which is a tedious case-by-case check. In light of the preceding results, we may assume that F has genus 4 and that S consists of $\alpha_1, \dots, \alpha_4$ together with 4 or 5 additional classes γ_i. We describe the system of γ_i by a matrix with 4 or 5 rows and 4 columns, consisting of entries 1, -1, or 0. (Conversely, any such matrix determines a collection of homology classes which one can try to realize.) We can normalize each such matrix according to the following principles: First of all we can arrange that the rows have monotonically nonincreasing size of support as one goes down the rows. Next, within any collection of rows with supports of the same size we can assume that the rows appear in lexicographical ordering according to the alphabet ordering $+1$, -1, 0. Next, by changing signs of the elements of S as required we can assume that the first nonzero element in each row and in each column is $+1$. A computer can easily crank out a list of all such matrices in lexicographical order. (It follows from the Summand Condition that there is at most one element of length 4 (i.e., involving all 4 classes α_i). Similarly, there are at most 2 elements of length 3. Again this follows from the Summand Condition, since two elements of length 3 must have two support elements in common and since the coefficients of the α_i appearing in the overlap of the supports of two classes must all have the same signs. The remaining classes must have support size 2.) At this point one should check the Summand Condition by checking that the determinant of every square submatrix is also $+1$, -1, or 0 and throw out those that do not meet this condition. Finally, in any particular case there may be extra symmetries at hand, exchanging pairs of rows or pairs of columns so as to produce a matrix higher up on our list. This last step is done by hand. Ultimately in this way we produce a list of 36 such matrices which one must show are realizable by actually drawing an appropriate planar diagram as above. (There is some redundancy in that some of the 4 by 4 matrices appear as submatrices of 5 by 4 matrices later in the list.) Although some of the required diagrams were a little difficult to discover, in the end all 36 were shown to be realizable. As just one example, here is one of the trickier realizable families:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

This corresponds to the family

$$S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_1 + \alpha_4, \alpha_2 - \alpha_3, \alpha_2 - \alpha_4\}$$

The following classes on the 4×2-punctured sphere lift the five composite classes:

$$A_1^+ + A_2^+ + A_3^+ + A_3^-, A_1^+ + A_3^+, A_1^+ + A_4^+, A_2^+ + A_3^-, A_2^- + A_4^+$$

This collection of classes can be realized by pairwise disjoint simple closed curves on the punctured sphere, and this realization then descends to give a realization of the given homology classes.

Discussion of the proof of Theorem 6, an algorithmic solution to the realizability problem. The results of Section 5 on realizing curves on a planar surface and of the first part of this Section 6, combine to provide a finite algorithm for realizing any family of homology classes by pairwise disjoint simple closed curves. As usual, let $S \subset H_1(F)$ denote a finite set of distinct homology classes satisfying the Intersection Condition, the Summand Condition, and the Size Condition of the the Main Theorem, which we wish to represent by pairwise disjoint simple closed curves. Suppose that the linear span of S has rank n and extract from S n elements $\alpha_1, \ldots, \alpha_n$ that form a basis for this span. Now each element γ_i in the remaining part of S can be expressed as a linear combination

$$\gamma_i = \sum_j \varepsilon_{ij}\, \alpha_j$$

in which we know by Lemma 6.1 that the coefficients satisfy $|\varepsilon_{ij}| \leq 1$. Moreover, we may also assume that we have already represented elements $\alpha_1, \ldots, \alpha_n$ by disjoint simple closed curves $A_1, \ldots, A_n$, by Proposition 4.2.

We attempt to represent the remaining classes in T, the complement of $\alpha_1, \dots, \alpha_n$ in S. Let $\widehat{F}$ denote F cut open along the A_i. Then $\widehat{F}$ is a 2-sphere with $2n$ holes (possibly with some additional handles attached, which play no role in the present discussion and which can safely be ignored), with orientable boundary consisting of one copy A_i^+ of each A_i and one copy A_i^- of each A_i with its orientation reversed. As in Section 5, the relevent homology $\mathcal{B} \subset H_1(\widehat{F})$ is generated by the homology classes of the boundary curves. Now the set S of homology classes can be realized by pairwise disjoint simple closed curves in F if and only if the classes in T can be lifted to a set $\widehat{T}$ of homology classes in $\mathcal{B} \subset H_1(\widehat{F})$ that can be realized by pairwise disjoint simple closed curves in $\widehat{F}$. Now, the classes γ_i have infinitely many pre-images in $H_1(\widehat{F})$, but only finitely many pre-images can be represented by simple closed curves, since by Lemma 5.1 the coefficients of the classes of the boundary curves must have absolute value at most 1, and all must have the same sign. There are only finitely many such lifts of each homology class and they may all be considered one-by-one for realizability using Proposition 5.3, which is itself finitely verifiable.

7. VARIOUS INSTRUCTIVE EXAMPLES

Here we present three interesting examples that point to some of the difficulties in finding necessary *and sufficient* conditions for realizability of a system of homology classes by pairwise disjoint simple closed curves. To start with we give an example showing that even when a system is realizable it is possible to get stuck, in the sense that one might realize all but one class and then have no way to realize the last class so as to be disjoint from the other curves. In particular, one might have to go back and change the curves already realized in order to complete the construction.

EXAMPLE 7.1. Non-extendable partial realizations of a realizable family of homology classes.

Let $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4\}$ be a system of homology classes on a surface of genus 4, in which $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is part of a standard symplectic basis. One can check that this collection satisfies all the necessary conditions in the Theorem 1. By Theorem 3, it is realizable by a system of pairwise disjoint simple closed curves. Explicitly, we can first realize $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ by standard curves A_1, A_2, A_3, A_4 in the surface F of genus 4. Letting $\widehat{F}$ denote the result of cutting F open along the A_i, we can then realize the remaining homology classes by lifting them to $H_1(\widehat{F})$, as follows:

$$A_1^+ + A_2^+, A_3^- + A_4^-, A_1^+ + A_2^+ + A_3^+, A_2^- + A_3^- + A_4^-$$

On the other hand the first three composite classes can also be realized using the lifts

$$A_1^+ + A_2^+, A_3^+ + A_4^+, A_1^- + A_2^- + A_3^-$$

This realization cannot be extended to a realization of the full collection, as one can see by case-by-case analysis. We remark that examples like this show that a strategy of aiming for "maximum disjointness" as one realizes the various curves will fail in general. One can similarly give simple examples showing that a strategy of "maximal nestedness" will also fail. For example, the lifts

$$A_1^+ + A_2^+, A_1^+ + A_1^- + A_2^+ + A_2^- + A_3^+ + A_4^+, A_1^+ + A_2^+ + A_3^+$$

again realize all but one class and this realization cannot be extended. In particular, this example also shows that one cannot simply use what one might call the "preferred" lifts of homology classes into $\widehat{F}$. That is, one may actually *require* extra terms of the form $A_i^+ + A_i^-$. (See the proof of Proposition 6.6, for examples.)

EXAMPLE 7.2. The Size Condition does not follow from the Intersection and Summand Conditions.

Let $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$ be a system of 10 homology classes on a surface of genus 4, in which $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is part of a standard symplectic basis, that is, is a basis for a summand of the homology on which the intersection pairing vanishes. In particular, the Size Condition is not satisfied. To check that this collection satisfies the Summand Condition holds we consider the 6 by 4 matrix whose rows are given by the last 6 "composite" classes expressed in terms of the first 4 classes:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

By brute force we can show that every square submatrix has determinant 0, 1, or -1, which gives the Summand Condition. Alternatively one can show directly that the transposed matrix describes 10 realizable homology classes in a surface of genus 6.

Here is an important example that satisfies the Intersection, Summand, and Size Conditions, but is not realizable.

THEOREM 7.3. *Let F be a surface of genus 5 and let*

$$\{\alpha_1, \ldots, \alpha_5, \beta_1, \ldots, \beta_5\}$$

be a standard symplectic basis for $H_1(F)$. Then the set

$$S = \{\alpha_1, \ldots, \alpha_5,\ \alpha_1 + \alpha_2 + \alpha_5,\ \alpha_1 + \alpha_2 + \alpha_3,\ \alpha_2 + \alpha_3 + \alpha_4,\ \alpha_3 + \alpha_4 - \alpha_5\}$$

satisfies the Intersection, Summand and Size Conditions, but cannot be realized by a corresponding collection of pairwise disjoint simple closed curves.

Proof. Since the intersection pairing of F vanishes on the subgroup of $H_1(F)$ generated by the α_i, the Intersection Condition clearly holds. One checks the Size Condition by observing that any set of 8 of the 9 classes including $\{\alpha_1, \ldots, \alpha_5\}$ can easily by realized, by Proposition 6.7. This implies the Size Condition for all subsets T of S not containing all 4 of the composite classes. But if a subset T does contain all 4 composite classes, then rank span $T \geq 4$ and card $T \leq 9 \leq 3$ rank span $T - 3$.

Here is the matrix of the composite classes expressed in terms of the first 5 independent classes.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}$$

One can visually check that all 2 by 2 minors have determinant 0 or ± 1. One can check by brute force that the same holds for the 4 by 4 and 3 by 3 minors. (A computer helps !) A better way is to check that all five 4 by 4 minors give realizable collections of homology classes. The best way is a neat trick : just observe that the transposed matrix, corresponding to $4+5=9$ classes on a surface of genus 4 is realizable by direct construction. This proves that the Summand Condition holds.

Suppose that this collection can in fact be realized by pairwise disjoint simple closed curves on F. If we cut open along the curve corresponding to

α_5 and cap off the resulting pair of boundary curves with two disks, then we have a realization of the corresponding collection

$$S' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_3 + \alpha_4\}$$

contracted down onto a surface of genus 4. This collection is definitely realizable. In particular we have a corresponding family of 4 pairwise disjoint simple closed curves on the 2×4-punctured sphere. Homology considerations show that the pair of disks, which must be removed, with the resulting boundaries identified, to re-construct the original surface of genus 5, both lie on the same side of the curves C_2 realizing $\gamma_2 = \alpha_1 + \alpha_2 + \alpha_3$ and C_3 realizing $\gamma_3 = \alpha_2 + \alpha_3 + \alpha_4$. But by bare hands one can show that for any realization of $\gamma_1 = \alpha_1 + \alpha_2$, γ_2, γ_3, and $\gamma_4 = \alpha_3 + \alpha_4$ by pairwise disjoint simple closed curves on the 2×4-punctured sphere, both of the curves C_1 and C_4 giving the classes γ_1 and γ_4 must be separated by both of the curves C_2 and C_3. But because our particular realization comes from a hypothesized realization of curves on a genus 5 surface, the added disks above must lie on the same side of C_2 as does C_1 *and also* as does C_4. This contradiction shows that the given collection cannot be realized.

REMARK. *The same set of 9 homology classes gives an example in* any *surface of genus greater than 5 of homology classes satisfying the Intersection, Summand, and Size Conditions that cannot be realized by a corresponding family of pairwise disjoint simple closed curves.*

This follows from Theorem 7.4 and Theorem 8.1 below.

8. SOME FINAL OBSERVATIONS

Notice that the Intersection, Summand, and Size Conditions in Theorem 1 make no mention of the genus of the ambient surface. A natural thought is that these three conditions might suffice to realize given homology classes by pairwise disjoint simple closed curves provided one is allowed to "stabilize" the surface by adding additional handles. Here we show that there is nothing gained by such stabilization.

PROPOSITION 8.1. *Suppose a surface F is expressed as a connected sum $F_1 \# F_2$ and we identify $H_1(F) = H_1(F_1) \oplus H_1(F_2)$ in the obvious way. Suppose further $S \subset H_1(F_1) \subset H_1(F)$ is a family of homology classes that can be realized by pairwise disjoint simple closed curves in F. Then S can be realized by pairwise disjoint simple closed curves in F_1.*

Proof. Suppose F_i has genus g_i, so that F has genus $g = g_1 + g_2$. As usual we let $S = \{\alpha_1, \dots, \alpha_n, \gamma_1, \dots, \gamma_k\}$, where $\alpha_1, \dots, \alpha_n$ form a basis for span S. In particular, $n \leq g_1$ and $\alpha_1, \dots, \alpha_n$ form part of a symplectic basis for the homology of F_1 as well as for F. We let $A_1, \dots, A_n, C_1, \dots, C_k$ denote pairwise disjoint simple closed curves representing the corresponding homology classes. We let $\widehat{F}$ denote the result of cutting F open along $A_1, \dots, A_n$ and $\overline{F}$ the result of filling in $\widehat{F}$ with $2n$ disks. Then $\overline{F}$ is a closed surface of genus $g - n$. We now view the curves $C_1, \dots, C_k$ as living in $\overline{F}$, but missing the added disks. Note that these curves are all null-homologous in $\overline{F}$ and hence each one of them separates $\overline{F}$ and $\widehat{F}$ into two pieces. The homology classes that the latter curves represent in the original surface F and in F_1 are determined up to sign by the collection of disks in $\overline{F}$ they surround. It follows that the curves $C_1, \dots, C_k$ all together separate $\overline{F}$ (or $\widehat{F}$) into $k+1$ pieces, with total genus $g - n$. In particular we see that there are $g - n$ homology classes $\alpha_{n+1}, \dots, \alpha_g$ represented by pairwise disjoint simple closed curves $A_{n+1}, \dots, A_g$ in $\overline{F}$ disjoint from the original $A_1, \dots, A_n$ and $C_1, \dots, C_k$ such that $\alpha_1, \dots, \alpha_g$ is half of a symplectic basis for the homology of F itself. It follows that if we surger away $A_{g_1+1}, \dots, A_g$, then $\alpha_1, \dots, \alpha_{g_1}$ represents half of a symplectic basis for the homology of the resulting surface F' of genus g_1, and if we identify the curves $A_1, \dots, A_{g_1}$ and $C_1, \dots, C_k$ with their images in F', we see that we have indeed embedded pairwise disjoint simple closed curves in $F' \cong F_1$ representing the corresponding homology classes. The point is that the homology classes γ_i of the C_i are completely determined as linear combinations of the α_j. And up to homeomorphism the curves $A_1, \dots, A_{g_1}$ are determined by representing a basis for a summand of the homology on which the intersection pairing vanishes.

The perspective developed above can also be applied to show that any system of pairwise disjoint homologically distinct simple closed curves can be expanded to a maximal set of $3g - 3$ such curves.

PROPOSITION 8.2. *Suppose that F is a closed, orientable surface of genus g and that S is a family of pairwise distinct homology classes represented by pairwise disjoint simple closed curves. Then S can be extended to a family of $3g - 3$ pairwise distinct homology classes represented by a set of pairwise disjoint simple closed curves, including the given collection of simple closed curves.*

Proof. As usual we let $S = \{\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_k\}$ be a given set of homology classes represented by a corresponding set of pairwise disjoint simple closed curves, $A_1, \ldots, A_n$, $C_1, \ldots, C_k$, where $\alpha_1, \ldots, \alpha_n$ form a basis for span S. We first argue that we can assume that $n = g$. If not, then as above some component of F cut open along all the given curves has positive genus. In that component we can then find a simple closed curve representing a homology class independent of those in S. In this way we increase the span of S until its rank is the maximum possible, namely g.

Now, when we cut open along our expanded family of simple closed curves all resulting components have genus 0. If all components have exactly three boundary components, then the euler characteristic argument of Section 2 shows that our collection already contains $3g - 3$ elements. Otherwise, some component G is a planar surface with at least $m \geq 4$ boundary components. Now when F is reconstructed starting from G one may think of attaching components of $F - G$ to G. None of these extra components can have just one boundary curve, since such a curve would be null-homologous. If such an extra component has two boundary curves, then the corresponding boundary curves of G would not be distinct, so we should actually be thinking in this case of simply identifying the two boundary curves. Suppose that some pair of boundary curves of G is identified in this way. Then it follows that in F the corresponding curve has a dual curve missing all the other curves representing elements of S. In particular that boundary curve of G represents a homology class in F independent of all the other classes in S. Now choose a simple closed curve in G that surrounds one of these two boundary curves and one other boundary curve. It follows that the corresponding homology class is distinct from all other elements of S. In this way we have again expanded the size of S.

Finally we may suppose no pair of boundary curves of G are to be identified. We want to claim that some simple closed curve in G surrounding 3 boundary curves is homologically nontrivial in F and homologically distinct from all other curves so far represented. A typical example of what we are up against is the following: Think of the surface of genus g expressed as the double of a $(g+1)$-holed sphere, with one side further decomposed by more pairwise disjoint, homologically distinct, simple closed curves. Now the challenge is to find more simple closed curves in the second side distinct from those already appearing in the first side. On the first side we have used at most $[3g - 3 - (g+1)]/2 = g - 1$ curves. But on the second side there are, for example, $(g+1)g/2$ different homology classes represented by simple closed curves surrounding just two boundary components.

So suppose G has $m > 3$ boundary curves. Then $H_1(G)$ is free abelian of rank $m - 1$, generated by the classes of the boundary curves, with the single relation that the sum of the classes of the boundary curves is 0. Consideration of the other components of $F - G$ then implies additional relations of the form "sum of boundary curves = 0" over the elements in each piece of a partition of the set of boundary components, where each partition piece has at least 3 elements. In particular we can obtain a basis for the image of $H_1(G)$ in $H_1(F)$ by selecting all but one boundary curve from each piece of the partition.

Now in such a surface as G with its m boundary components there are at most $m - 3$ pairwise disjoint simple closed curves, pairwise homologically distinct and homologically distinct from the boundary curves. Even if G were not a planar surface, there would be at most $m - 3$ such curves homologous to some linear combination of the boundary curves. If the components of $F - G$ are $G_1, \ldots, G_r$, where G_i has m_i boundary curves, then $\sum_{i=1}^{r} m_i = m$. Moreover, the image of $H_1(G)$ in $H_1(F)$ has a basis of $\sum(m_i - 1) = m - r$ elements. Note also that $1 \leq r \leq m/3$, since no component G_i should have just one or two boundary curves. Now in G_i there are at most $m_i - 3$ pairwise disjoint simple closed curves representing homology classes in the linear span of the classes represented by the boundary curves of G_i. It follows that there are already in the originally given collection of curves at most $\sum(m_i - 3) + m = 2m - 3r$ distinct homology classes. On the other hand, within G itself there are some $2^{m-r} - m - 1$ homology classes represented by simple closed curves, excluding the classes represented by the boundary curves and the 0 class. Therefore, in order to expand our originally given collection of simple closed curves by adding a curve inside G, we need to have

$$2^{m-r} - m - 1 > 2m - 3r$$

or

$$2^x - 3x - 1 > 0$$

where $x = m - r$. But $2^x - 3x - 1 \leq 0$ only for $x = 1$ or $x = 2$ (among integral x). That is to say there is trouble only if $m - r = 1$ or $m - r = 2$, i.e., $r = m - 1$ or $m - 2$. But we already noted that we have $1 \leq r \leq m/3$. So, $m - 2 \leq r \leq m/3$, which implies that $m \leq 3$. But we had already seen that we could assume $m > 3$. Thus there must be suitable simple closed curves in G that can be added to the given collection while maintaining the required homological distinctness.

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