2. Necessary conditions

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We conclude by proving an analogue of this for homologically nontrivial and distinct curves.

THEOREM 7. *Let F be a closed orientable surface of genus $g \geq 2$, and let $S \subset H_1(F)$ be a set of pairwise distinct nonzero homology classes represented by a corresponding family of pairwise disjoint simple closed curves in F. Then this family of simple closed curves can be extended to a family of $3g-3$ pairwise disjoint simple closed curves in F representing nontrivial, pairwise distinct homology classes in $H_1(F)$.*

Here is a summary of the contents of the rest of the paper: Section 2 contains the proof of Theorem 1 deriving the fundamental necessary conditions. Sections 3 and 4 deal with the cases of one homology class and with independent homology classes, and provide two proofs of Theorem 2. In Section 5 we give an analysis of simple closed curves on a planar surface. In Section 6 there is the proof of the main positive realizability statement, Theorem 3, ending with a discussion of Theorem 6. In Section 7 we present several examples that illustrate some of the subtleties involved in finding a more complete and definitive result than that given here, including the nonrealizability result stated as Theorem 4. Finally in Section 8 we give the proofs of Theorem 5 and 7.

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2. NECESSARY CONDITIONS

It is quite clear that the Intersection Condition must hold, since the intersection number of two disjoint 1-cycles is necessarily 0. The necessity of the Summand Condition follows immediately from the following lemma.

LEMMA 2.1. *Let F be a closed orientable surface of genus $g \geq 1$ and let $S \subset H_1(F)$ be a set of pairwise distinct homology classes represented by a corresponding set of pairwise disjoint simple closed curves in F. Then the image of S spans a direct summand of $H_1(F)$.*

Proof. Let $A \subset F$ be the union of the simple closed curves representing the elements of S in $H_1(F)$. Consider the long exact homology sequence of the pair (F, A).

$$\cdots \to H_1(A) \to H_1(F) \to H_1(F, A) \to \cdots$$

Now the linear span of S in $H_1(F)$ is identified with the image of $H_1(A)$ in $H_1(F)$. But $H_1(F, A)$ is free (by Poincaré Duality), so we see that the image of $H_1(A)$ is a direct summand, as required.

The following result gives the Size Condition. The construction described in the proof below will be important, as it describes an effective way to approach the problem of explicitly realizing a system of pairwise disjoint curves.

LEMMA 2.2. *Let F be a closed orientable surface of genus $g \geq 1$, let $S \subset H_1(F)$ be a set of pairwise distinct homology classes represented by a corresponding set of pairwise disjoint simple closed curves in F, and let $n =$* rank span S. *Then* card $S \leq \max\{3n - 3, 1\}$.

Proof. If $n = 1$, then it follows from Lemma 2.1 that S must consist of a single element, and the desired inequality trivially holds. Henceforth we assume that $n > 1$. The proof in this case will amount to cutting the surface up into pieces along the given simple closed curves and using the pieces to calculate the euler characteristic of the surface. It is easy to see that $g \geq n$. We will first assume that $g = n$. At the end we will indicate how to modify the proof to handle the case $g > n$.

Let $\alpha_1, \ldots, \alpha_n \in S$ form a basis for span S and let $A_1, \ldots, A_n$ be the corresponding disjoint simple closed curves in F. Let $\gamma_1, \ldots, \gamma_m \in S$ be the remaining elements of S and $C_1, \ldots, C_m$ be the corresponding disjoint simple closed curves in F. Let $\widehat{F}$ denote the surface F cut open along the A_i. Then $\widehat{F}$ is a connected, orientable surface and has $2n$ boundary curves and genus $g - n = 0$. Note that $\chi(\widehat{F}) = \chi(F)$ by the sum formula for euler characteristics. In $\widehat{F}$ each of the $m =$ card $S - n$ curves C_j is homologous to a sum of boundary curves, with multiplicities ± 1, since C_j does not separate F, but does separate $\widehat{F}$. Now $\overline{F} = \widehat{F} - \cup C_j$ still has genus 0 and consists of $m + 1$ planar components X_ℓ. Again note that $\chi(\overline{F}) = \chi(\widehat{F})$. No X_ℓ can be a disk, since otherwise its boundary curve would be nullhomologous in F. Similarly, no X_ℓ can be an annulus, since otherwise, the two boundary curves, belonging to the original collection of curves would represent the same homology class in F, up to sign. It follows, therefore, from the classification of surfaces, that each X_ℓ has Euler characteristic ≤ -1. Therefore, when $n > 1$ and $n = g$,

$$\chi(F) = \chi(\bar{F}) = 2 - 2n = \sum \chi(X_\ell) \leq (\text{card } S - n + 1)(-1)$$

or equivalently card $S \leq 3n - 3$, as required.

It remains to consider the case when $g > n > 1$. In this case, we first proceed as before, cutting open along the A_i, obtaining a connected surface $\widehat{F}$ of genus $g - n$ and with $2n$ boundary curves, containing the m curves C_j, each of which is homologous to a sum of boundary curves in $\widehat{F}$. Now each of the C_j separates $\widehat{F}$, and we may further cut open along the C_j, obtaining a surface with $m + 1$ components and total genus $g - n$. It follows that there are additional pairwise disjoint simple closed curves E_k, $k = 1, \ldots, g - n$, in $\widehat{F}$, reducing $\widehat{F}$ to a planar surface of genus 0 when we cut open along the E_k and cap off the resulting $2(g - n)$ boundary curves with disks. Call this latter surface $\bar{F}$, topologically a 2-sphere with $2n$ holes. Now the C_j separate $\bar{F}$ into $m + 1 =$ card $S - n + 1$ planar components X_ℓ. As before, each X_ℓ has Euler characteristic ≤ -1. Therefore, again,

$$\chi(\bar{F}) = 2 - 2n = \sum \chi(X_\ell) \leq (\text{card } S - n + 1)(-1)$$

or equivalently card $S \leq 3n - 3$, as required.

3. Sufficiency for a single homology class

Here we collect some basic information about the embedding of a single simple closed curve in a surface, and offer an alternative, elementary proof of Theorem 2 for the well-known case of a single homology class.

LEMMA 3.1. *A nonzero homology class $\alpha \in H_1(F)$ is primitive if and only if there exists $\gamma \in H_1(F)$ such that $\gamma \cdot \alpha = 1$.*

Proof. A nonzero element of a finitely generated free abelian group is primitive if and only if it is part of a basis if and only if there is a $\mathbf{Z}$-valued homomorphism that takes the value 1 on it. Recall that taking intersection numbers of 1-cycles defines a skew symmetric bilinear form on $H_1(F)$. The content of Poincaré Duality in this situation is that this bilinear form is nonsingular, that is, the adjoint homomorphism $H_1(F) \to \mathrm{Hom}\big(H_1(F), \mathbf{Z}\big)$ is an isomorphism. The lemma then follows.