7. Various instructive examples

We attempt to represent the remaining classes in T, the complement of $\alpha_1, \ldots, \alpha_n$ in S. Let $\widehat{F}$ denote F cut open along the A_i. Then $\widehat{F}$ is a 2-sphere with $2n$ holes (possibly with some additional handles attached, which play no role in the present discussion and which can safely be ignored), with orientable boundary consisting of one copy A_i^+ of each A_i and one copy A_i^- of each A_i with its orientation reversed. As in Section 5, the relevent homology $\mathcal{B} \subset H_1(\widehat{F})$ is generated by the homology classes of the boundary curves. Now the set S of homology classes can be realized by pairwise disjoint simple closed curves in F if and only if the classes in T can be lifted to a set $\widehat{T}$ of homology classes in $\mathcal{B} \subset H_1(\widehat{F})$ that can be realized by pairwise disjoint simple closed curves in $\widehat{F}$. Now, the classes γ_i have infinitely many pre-images in $H_1(\widehat{F})$, but only finitely many pre-images can be represented by simple closed curves, since by Lemma 5.1 the coefficients of the classes of the boundary curves must have absolute value at most 1, and all must have the same sign. There are only finitely many such lifts of each homology class and they may all be considered one-by-one for realizability using Proposition 5.3, which is itself finitely verifiable.

7. Various instructive examples

Here we present three interesting examples that point to some of the difficulties in finding necessary *and sufficient* conditions for realizability of a system of homology classes by pairwise disjoint simple closed curves. To start with we give an example showing that even when a system is realizable it is possible to get stuck, in the sense that one might realize all but one class and then have no way to realize the last class so as to be disjoint from the other curves. In particular, one might have to go back and change the curves already realized in order to complete the construction.

Example 7.1. Non-extendable partial realizations of a realizable family of homology classes.

Let $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1+\alpha_2, \alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3, \alpha_2+\alpha_3+\alpha_4\}$ be a system of homology classes on a surface of genus 4, in which $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is part of a standard symplectic basis. One can check that this collection satisfies all the necessary conditions in the Theorem 1. By Theorem 3, it is realizable by a system of pairwise disjoint simple closed curves. Explicitly, we can first realize $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ by standard curves A_1, A_2, A_3, A_4 in the surface F of genus 4. Letting $\widehat{F}$ denote the result of cutting F open along the A_i, we can then realize the remaining homology classes by lifting them to $H_1(\widehat{F})$, as follows:

$$A_1^+ + A_2^+, A_3^- + A_4^-, A_1^+ + A_2^+ + A_3^+, A_2^- + A_3^- + A_4^-$$

On the other hand the first three composite classes can also be realized using the lifts

$$A_1^+ + A_2^+, A_3^+ + A_4^+, A_1^- + A_2^- + A_3^-$$

This realization cannot be extended to a realization of the full collection, as one can see by case-by-case analysis. We remark that examples like this show that a strategy of aiming for "maximum disjointness" as one realizes the various curves will fail in general. One can similarly give simple examples showing that a strategy of "maximal nestedness" will also fail. For example, the lifts

$$A_1^+ + A_2^+, A_1^+ + A_1^- + A_2^+ + A_2^- + A_3^+ + A_4^+, A_1^+ + A_2^+ + A_3^+$$

again realize all but one class and this realization cannot be extended. In particular, this example also shows that one cannot simply use what one might call the "preferred" lifts of homology classes into $\widehat{F}$. That is, one may actually *require* extra terms of the form $A_i^+ + A_i^-$. (See the proof of Proposition 6.6, for examples.)

EXAMPLE 7.2. The Size Condition does not follow from the Intersection and Summand Conditions.

Let $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$ be a system of 10 homology classes on a surface of genus 4, in which $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is part of a standard symplectic basis, that is, is a basis for a summand of the homology on which the intersection pairing vanishes. In particular, the Size Condition is not satisfied. To check that this collection satisfies the Summand Condition holds we consider the 6 by 4 matrix whose rows are given by the last 6 "composite" classes expressed in terms of the first 4 classes:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

By brute force we can show that every square submatrix has determinant 0, 1, or -1, which gives the Summand Condition. Alternatively one can show directly that the transposed matrix describes 10 realizable homology classes in a surface of genus 6.

Here is an important example that satisfies the Intersection, Summand, and Size Conditions, but is not realizable.

THEOREM 7.3. *Let F be a surface of genus 5 and let*

$$\{\alpha_1, \dots, \alpha_5, \beta_1, \dots, \beta_5\}$$

be a standard symplectic basis for $H_1(F)$. Then the set

$$S = \{\alpha_1, \dots, \alpha_5,\ \alpha_1 + \alpha_2 + \alpha_5,\ \alpha_1 + \alpha_2 + \alpha_3,\ \alpha_2 + \alpha_3 + \alpha_4,\ \alpha_3 + \alpha_4 - \alpha_5\}$$

satisfies the Intersection, Summand and Size Conditions, but cannot be realized by a corresponding collection of pairwise disjoint simple closed curves.

Proof. Since the intersection pairing of F vanishes on the subgroup of $H_1(F)$ generated by the α_i, the Intersection Condition clearly holds. One checks the Size Condition by observing that any set of 8 of the 9 classes including $\{\alpha_1, \dots, \alpha_5\}$ can easily by realized, by Proposition 6.7. This implies the Size Condition for all subsets T of S not containing all 4 of the composite classes. But if a subset T does contain all 4 composite classes, then rank span $T \geq 4$ and card $T \leq 9 \leq 3$ rank span $T - 3$.

Here is the matrix of the composite classes expressed in terms of the first 5 independent classes.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 \end{pmatrix}$$

One can visually check that all 2 by 2 minors have determinant 0 or ± 1. One can check by brute force that the same holds for the 4 by 4 and 3 by 3 minors. (A computer helps !) A better way is to check that all five 4 by 4 minors give realizable collections of homology classes. The best way is a neat trick : just observe that the transposed matrix, corresponding to $4+5=9$ classes on a surface of genus 4 is realizable by direct construction. This proves that the Summand Condition holds.

Suppose that this collection can in fact be realized by pairwise disjoint simple closed curves on F. If we cut open along the curve corresponding to

α_5 and cap off the resulting pair of boundary curves with two disks, then we have a realization of the corresponding collection

$$S' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_3 + \alpha_4\}$$

contracted down onto a surface of genus 4. This collection is definitely realizable. In particular we have a corresponding family of 4 pairwise disjoint simple closed curves on the 2×4-punctured sphere. Homology considerations show that the pair of disks, which must be removed, with the resulting boundaries identified, to re-construct the original surface of genus 5, both lie on the same side of the curves C_2 realizing $\gamma_2 = \alpha_1 + \alpha_2 + \alpha_3$ and C_3 realizing $\gamma_3 = \alpha_2 + \alpha_3 + \alpha_4$. But by bare hands one can show that for any realization of $\gamma_1 = \alpha_1 + \alpha_2$, γ_2, γ_3, and $\gamma_4 = \alpha_3 + \alpha_4$ by pairwise disjoint simple closed curves on the 2×4-punctured sphere, both of the curves C_1 and C_4 giving the classes γ_1 and γ_4 must be separated by both of the curves C_2 and C_3. But because our particular realization comes from a hypothesized realization of curves on a genus 5 surface, the added disks above must lie on the same side of C_2 as does C_1 *and also* as does C_4. This contradiction shows that the given collection cannot be realized.

REMARK. *The same set of 9 homology classes gives an example in* any *surface of genus greater than 5 of homology classes satisfying the Intersection, Summand, and Size Conditions that cannot be realized by a corresponding family of pairwise disjoint simple closed curves.*

This follows from Theorem 7.4 and Theorem 8.1 below.

8. Some Final Observations

Notice that the Intersection, Summand, and Size Conditions in Theorem 1 make no mention of the genus of the ambient surface. A natural thought is that these three conditions might suffice to realize given homology classes by pairwise disjoint simple closed curves provided one is allowed to "stabilize" the surface by adding additional handles. Here we show that there is nothing gained by such stabilization.

PROPOSITION 8.1. *Suppose a surface F is expressed as a connected sum $F_1 \# F_2$ and we identify $H_1(F) = H_1(F_1) \oplus H_1(F_2)$ in the obvious way. Suppose further $S \subset H_1(F_1) \subset H_1(F)$ is a family of homology classes that can be realized by pairwise disjoint simple closed curves in F. Then S can be realized by pairwise disjoint simple closed curves in F_1.*