# 2. DEFINITION OF $L^2$ -COHOMOLOGY


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## 2. DEFINITION OF $L^2$-COHOMOLOGY

Let $M$ be as above. Let $\Lambda^p(M)$ denote the Hilbert space of square-integrable $p$-forms on $M$. The completeness of $M$ enters in one crucial way, in allowing us to integrate by parts on $M$ in the sense of the following lemma.

LEMMA 1 (Gaffney [13]). *Suppose that $\omega$, $\eta$, $d\omega$ and $d\eta$ are smooth square-integrable differential forms on $M$. Then*

$$\int_M d\omega \wedge \eta + (-1)^{deg(\omega)} \int_M \omega \wedge d\eta = 0. \tag{2.1}$$

*Proof.* We claim that there is a sequence $\{\phi_i\}_{i=1}^{\infty}$ of compactly-supported functions on $M$ with the properties that

1. There is a constant $C > 0$ such that for all $i$ and almost all $m \in M$, $|\phi_i(m)| \leq C$ and $|d\phi_i(m)| \leq C$.
2. For almost all $m \in M$, $\lim_{i\to\infty} \phi_i(m) = 1$ and $\lim_{i\to\infty} |d\phi_i(m)| = 0$.

To construct the sequence $\{\phi_i\}_{i=1}^{\infty}$, let $m_0$ be a basepoint in $M$. Let $f \in C_0^{\infty}([0,\infty))$ be a nonincreasing function such that if $x \in [0,1]$ then $f(x) = 1$. Put $\phi_i(m) = f\left(\frac{1}{i} d(m_0, m)\right)$. This gives the desired sequence. The completeness of $M$ ensures that $\phi_i$ is compactly-supported. Note that $\phi_i$ is *a priori* only a Lipschitz function, but this is good enough for our purposes.

Using Lebesgue Dominated Convergence and the fact that we can integrate by parts for compactly-supported forms, we have

$$\begin{aligned} \int_M d\omega \wedge \eta + (-1)^{deg(\omega)} \int_M \omega \wedge d\eta &= \int_M d(\omega \wedge \eta) \\ &= \lim_{i\to\infty} \int_M \phi_i \, d(\omega \wedge \eta) \\ &= - \lim_{i\to\infty} \int_M d\phi_i \wedge \omega \wedge \eta = 0. \end{aligned} \tag{2.2}$$

This proves the lemma. □

Let $d^*$ be the formal adjoint to $d$. Using Lemma 1, one can construct a self-adjoint operator $\triangle = dd^* + d^*d$ acting on $\Lambda^*(M)$, with domain

$$\begin{aligned} \mathrm{Dom}(\triangle) = &\{\omega \in \Lambda^*(M) : d\omega,\ d^*\omega,\ dd^*\omega \\ &\text{and } d^*d\omega \text{ are square-integrable}\}. \end{aligned} \tag{2.3}$$

Let $\triangle_p$ denote the restriction of $\triangle$ to $\Lambda^p(M)$. The spectrum $\sigma(\triangle_p)$ of $\triangle_p$ is a closed subset of $[0,\infty)$.

LEMMA 2. *The kernel of* $\triangle_p$ *is* $\{\omega \in \Lambda^p(M) : d\omega = d^*\omega = 0\}$.

*Proof.* Clearly $\{\omega \in \Lambda^p(M) \colon d\omega = d^*\omega = 0\} \subseteq \mathrm{Ker}(\triangle_p)$. If $\omega \in \mathrm{Ker}(\triangle_p)$ then by elliptic regularity, $\omega$ is smooth. Using integration by parts,

$$0 = \langle \omega, \triangle_p \omega \rangle = \langle d\omega, d\omega \rangle + \langle d^*\omega, d^*\omega \rangle, \text{ so } d\omega = d^*\omega = 0. \qquad \square$$

WARNING. Unlike what happens with compact manifolds, it is possible that $\mathrm{Ker}(\triangle_p) = 0$ but nevertheless $0 \in \sigma(\triangle_p)$. The simplest example of this is when $M = \mathbf{R}$ and $p = 0$. By Lemma 2, $\mathrm{Ker}(\triangle_0)$ consists of square-integrable functions $f$ on $\mathbf{R}$ such that $df = 0$. Clearly the only such function is the zero function. However, under Fourier transform, $\triangle_0$ is equivalent to the multiplication operator by $k^2$ on $L^2(\mathbf{R})$ and hence $\sigma(\triangle_0) = [0, \infty)$.

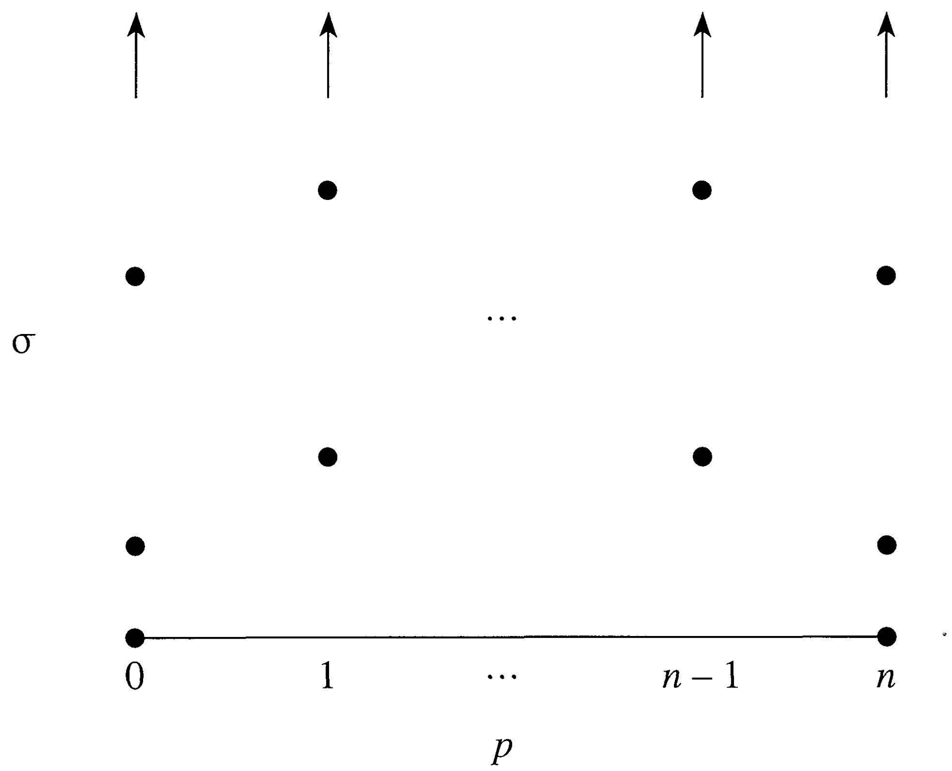


FIGURE 1

EXAMPLES. We now give $\sigma(\triangle_p)$ for simply-connected space forms.

1. $M$ is the standard sphere $S^n$. From [14],

$$\sigma(\triangle_p) = \{(k+p)(k+n+1-p)\}_{k=0}^{\infty} \cup \{(k+p+1)(k+n-p)\}_{k=0}^{\infty}. \tag{2.4}$$

(See Fig. 1.) The details of the spectrum are not important for us. We only wish to note that $\sigma(\triangle_p)$ is discrete, and $0 \in \sigma(\triangle_p)$ if $p = 0$ or $p = n$. These statements are a consequence of the fact that $M$ is closed. Namely, if $M^n$ is any closed Riemannian manifold then $\sigma(\triangle_p)$ is discrete and $\mathrm{Ker}(\triangle_p) \cong \mathrm{H}^p(M; \mathbf{C})$. In particular, $\mathrm{Ker}(\triangle_0) \cong \mathrm{H}^0(M; \mathbf{C}) = \mathbf{C}$ consists of the constant functions and $\mathrm{Ker}(\triangle_n) \cong \mathrm{H}^n(M; \mathbf{C}) = \mathbf{C}$ consists of multiples of the volume form.

2. $M$ is the standard Euclidean space $\mathbf{R}^n$. As the $p$-forms on $\mathbf{R}^n$ consist of $\binom{n}{p}$ copies of the functions, it is enough to consider $\sigma(\triangle_0)$. By Fourier analysis, $\sigma(\triangle_0) = [0, \infty)$. Thus $\sigma(\triangle_p) = [0, \infty)$ for all $0 \le p \le n$. (See Fig. 2.) Note that $\mathrm{Ker}(\triangle_p) = 0$ for all $p$.

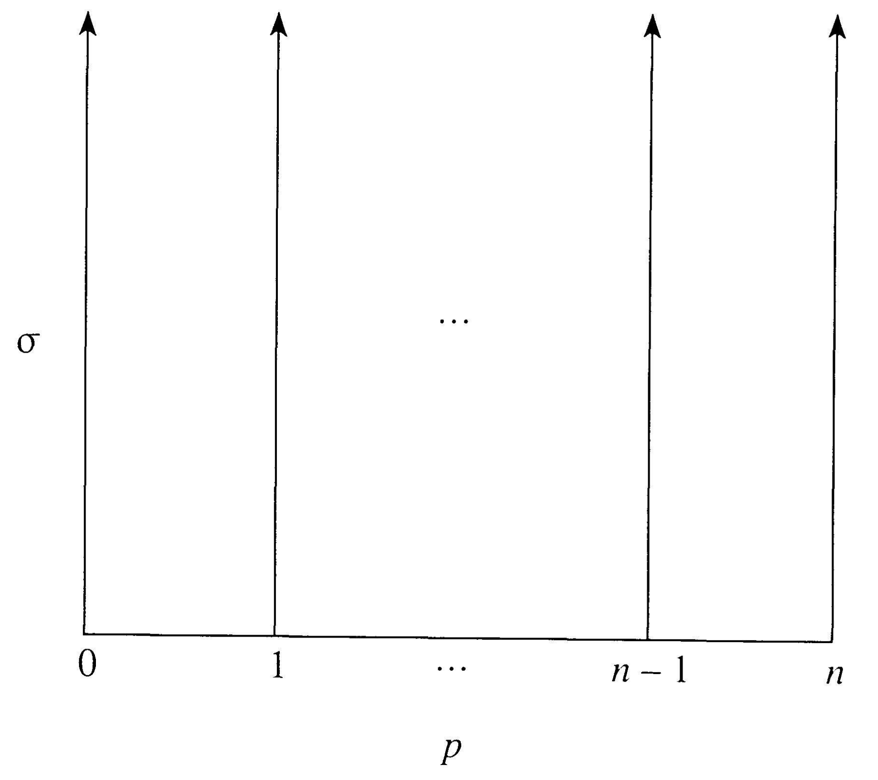


FIGURE 2

3. $M$ is the hyperbolic space $H^{2n}$. From [9],

$$\sigma(\triangle_p) = \begin{cases} \left[\frac{(2n-2p-1)^2}{4}, \infty\right) & \text{if } 0 \le p \le n-1, \\ \{0\} \cup \left[\frac{1}{4}, \infty\right) & \text{if } p = n, \\ \left[\frac{(2p-2n-1)^2}{4}, \infty\right) & \text{if } n+1 \le p \le 2n. \end{cases}$$

(See Fig. 3.) There is an infinite-dimensional kernel to $\triangle_n$. Otherwise, the spectrum is strictly bounded away from zero.

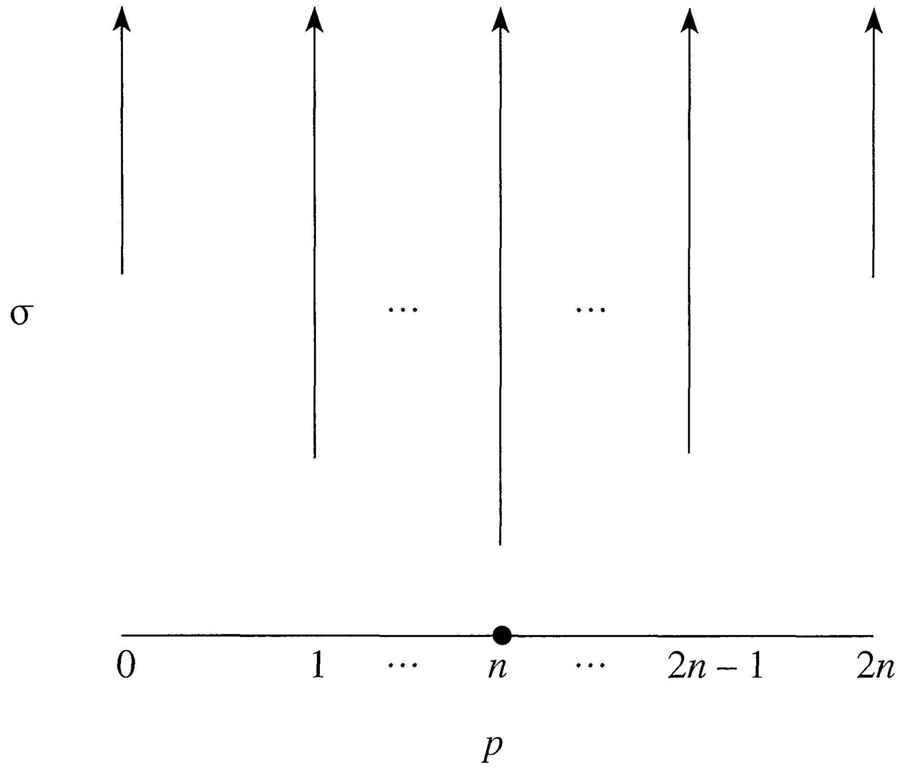


FIGURE 3

4. $M$ is the hyperbolic space $H^{2n+1}$. From [9],

$$\sigma(\triangle_p) = \begin{cases} \left[\frac{(2n-2p)^2}{4}, \infty\right) & \text{if } 0 \le p \le n, \\ \left[\frac{(2p-2n-2)^2}{4}, \infty\right) & \text{if } n+1 \le p \le 2n+1. \end{cases}$$

(See Fig. 4.) For all $p$, $\mathrm{Ker}(\triangle_p) = 0$. The continuous spectrum extends down to zero in degrees $n$ and $n+1$, and is strictly bounded away from zero in other degrees.

Comparing Figures 1-4, the spectra do not have much in common. However, one common feature is that zero lies in $\sigma(\triangle_p)$ for some $p$, although for different reasons in the different cases. In Figure 1, it is because $\triangle_0$ has a nonzero finite-dimensional kernel. In Figure 2, it is because zero lies in the continuous spectrum of $\triangle_p$ for all $p$. In Figure 3, it is because $\triangle_n$ has an infinite-dimensional kernel. And in Figure 4, it is because zero lies in the continuous spectrum of $\triangle_p$ for $p = n$ and $p = n+1$.

The above examples, along with others, motivate the zero-in-the-spectrum question. One can pose the question for various classes of manifolds, such as

1. Complete Riemannian manifolds.
2. Complete Riemannian manifolds of bounded geometry, meaning that the injectivity radius is positive and the sectional curvature $K$ satisfies $|K| \leq 1$.
3. Uniformly contractible Riemannian manifolds, meaning that for all $r > 0$, there is an $R(r) \geq r$ such that for all $m \in M$, the metric ball $B_r(m)$ can be contracted to a point within $B_{R(r)}(m)$.
4. Universal covers of closed Riemannian manifolds.
5. Universal covers of closed aspherical Riemannian manifolds.

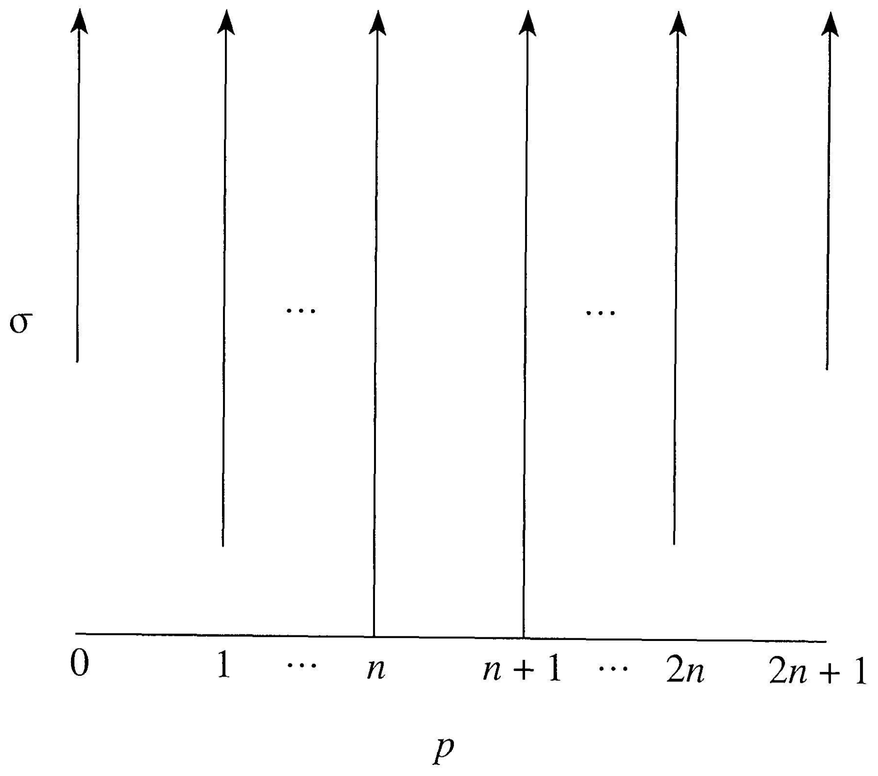


FIGURE 4

There are obvious inclusions $\begin{array}{ccccc} 5 & \subset & 4 & \subset & 2 \\ \cap & & & & \cap \\ 3 & & \subset & & 1 \end{array}$. As we shall discuss, there are some reasons to believe that the answer to the zero-in-the-spectrum question is "yes" in class 5, but the evidence for a "yes" answer in class 1 consists mainly of a lack of counterexamples.

In order to make the study of the spectrum of $\triangle_p$ more precise, the Hodge decomposition

$$\Lambda^p(M) = \mathrm{Ker}(\triangle_p) \oplus \overline{\mathrm{Im}(d)} \oplus \Lambda^p(M)/\,\mathrm{Ker}(d) \tag{2.5}$$

is useful. The operator $\triangle_p$ decomposes with respect to (2.5) as a direct sum of three operators. If we know the spectrum of the Laplace-Beltrami operator on all forms of degree less than $p$ then the new information in degree $p$ consists of $\mathrm{Ker}(\triangle_p)$ and the spectrum of $\triangle_p$ on $\Lambda^p(M)/\mathrm{Ker}(d)$. So we can ask the more precise questions:

1. What is $\dim\big(\mathrm{Ker}(\triangle_p)\big)$ ?
2. Is zero in $\sigma\left(\triangle_p \text{ on } \Lambda^p(M)/\mathrm{Ker}(d)\right)$ ?

By its definition, $\triangle_p$ involves the first derivatives of the metric tensor. We now show that the answer to the zero-in-the-spectrum question only depends on the $C^0$-properties of the metric tensor. To do so, we reformulate the question in terms of $L^2$-cohomology. Define a subspace $\Omega^p(M)$ of $\Lambda^p(M)$ by

$$\Omega^p(M) = \{\omega \in \Lambda^p(M) : d\omega \text{ is square-integrable}\}, \tag{2.6}$$

where $d\omega$ is initially interpreted in a distributional sense. The subspace $\Omega^p(M)$ is cooked up so that we have a cochain complex

$$\ldots \xrightarrow{d_{p-1}} \Omega^p(M) \xrightarrow{d_p} \Omega^{p+1}(M) \xrightarrow{d_{p+1}} \ldots \tag{2.7}$$

LEMMA 3. $\mathrm{Ker}(d_p)$ *is a subspace of* $\Omega^p(M)$ *which is closed in* $\Lambda^p(M)$.

*Proof.* Suppose that $\{\eta_i\}_{i=1}^{\infty}$ is a sequence in $\mathrm{Ker}(d_p)$ which converges to $\omega \in \Lambda^p(M)$ in an $L^2$-sense. We must show that the distributional form $d\omega$ vanishes. Given a smooth compactly-supported $(p+1)$-form $\rho$, we have

$$\langle d\omega, \rho\rangle = \langle \omega, d^*\rho\rangle = \lim_{i\to\infty}\langle \eta_i, d^*\rho\rangle = \lim_{i\to\infty}\langle d\eta_i, \rho\rangle = 0\,. \tag{2.8}$$

The lemma follows. □

DEFINITION 1. *The p-th unreduced $L^2$-cohomology group of M is* $\mathrm{H}^p_{(2)}(M) = \mathrm{Ker}(d_p)/\mathrm{Im}(d_{p-1})$. *The p-th reduced $L^2$-cohomology group of M is* $\overline{\mathrm{H}}^p_{(2)}(M) = \mathrm{Ker}(d_p)/\overline{\mathrm{Im}(d_{p-1})}$, *a Hilbert space.*

The square-integrability condition on the forms should be thought of as a global decay condition, not as a local regularity condition. One can also compute $\mathrm{H}^*_{(2)}(M)$ using a complex as in (2.7) where the forms are additionally required to be smooth [20, Prop. 9].

There is an obvious surjection $i_p : \mathrm{H}^p_{(2)}(M) \to \overline{\mathrm{H}}^p_{(2)}(M)$. Clearly $i_p$ is an isomorphism if and only if $d_{p-1}$ has closed image.

PROPOSITION 1.

1. $\operatorname{Ker}(\triangle_p) \cong \overline{\mathrm{H}}^p_{(2)}(M)$.
2. $0 \notin \sigma\left(\triangle_p \text{ on } \Lambda^p(M)/\operatorname{Ker}(d)\right)$ *if and only if* $i_{p+1}$ *is an isomorphism.*

*Proof.* 1. Using Lemma 2, we have

$$\begin{aligned}\operatorname{Ker}(\triangle_p) &= \{\omega \in \Lambda^p(M) : d\omega = d^*\omega = 0\} \\ &= \operatorname{Ker}(d_p) \cap \overline{\operatorname{Im}(d_{p-1})}^{\perp} \cong \overline{\mathrm{H}}^p_{(2)}(M).\end{aligned} \tag{2.9}$$

The first part of the proposition follows.

2. Suppose first that $\triangle_p$ has a bounded inverse on $\Lambda^p(M)/\operatorname{Ker}(d)$. Given $\mu \in \Lambda^p(M)$, let $\overline{\mu}$ denote its class in $\Lambda^p(M)/\operatorname{Ker}(d)$. Define an operator $S$ on smooth compactly-supported $(p+1)$-forms by $S(\omega) = d\triangle_p^{-1}\overline{d^*\omega}$. Then $S$ extends to a bounded operator on $\Lambda^{p+1}(M)$. Let $\{\eta_i\}_{i=1}^{\infty}$ be a sequence in $\Omega^p(M)$ such that $\lim_{i\to\infty} d\eta_i = \omega$ for some $\omega \in \Lambda^{p+1}(M)$. Then for each $i$, we have $d\eta_i = S(d\eta_i)$ and so $\omega = S(\omega)$. Thus $\omega \in \operatorname{Im}(d)$ and so $\operatorname{Im}(d)$ is closed.

Now suppose that $\triangle_p$ does not have a bounded inverse on $\Lambda^p(M)/\operatorname{Ker}(d)$. Then there is a sequence of positive numbers $r_1 > s_1 > r_2 > s_2 > \ldots$ tending towards zero and an orthonormal sequence $\{\eta_i\}_{i=1}^{\infty}$ in $\Lambda^p(M)/\operatorname{Ker}(d)$ such that with respect to the spectral projection $P$ of $\triangle_p$ (acting on $\Lambda^p(M)/\operatorname{Ker}(d)$), $\eta_i \in \operatorname{Im}\left(P\left([s_i, r_i]\right)\right)$. Put $\lambda_i = \|d\eta_i\|$. Then $\lim_{i\to\infty} \lambda_i = 0$. Let $\{c_i\}_{i=1}^{\infty}$ be a sequence in $\mathbf{R}^+$ such that $\sum_{i=1}^{\infty} c_i^2 = \infty$ and $\sum_{i=1}^{\infty} c_i\lambda_i < \infty$. Put $\omega = \sum_{i=1}^{\infty} c_i d\eta_i$. Then $\omega \in \overline{\operatorname{Im}(d)}$. Suppose that $\omega = d\mu$ for some $\mu \in \Omega^p(M)$. By the spectral theorem, we must have $\overline{\mu} = \sum_{i=1}^{\infty} c_i\eta_i$. However, this is not square-integrable. Thus $\operatorname{Im}(d)$ is not closed. The proposition follows. □

COROLLARY 1. *Zero does not lie in* $\sigma(\triangle_p)$ *for any* $p$ *if and only if* $\mathrm{H}^p_{(2)}(M) = 0$ *for all* $p$*, i.e. if the complex (2.7) is contractible.*

So a counterexample to the zero-in-the-spectrum question would consist of a manifold $M$ whose complex (2.7) is contractible. By way of comparison, recall that the compactly-supported complex-valued cohomology of $M$ is computed by a cochain complex similar to (2.7), except using compactly-supported smooth forms. As $\mathrm{H}_c^{dim(M)}(M;\mathbf{C}) \neq 0$, this latter complex is never contractible. And the ordinary complex-valued cohomology of $M$ is computed by a cochain complex similar to (2.7), except using smooth forms without any decay conditions. Again, as $\mathrm{H}^0(M;\mathbf{C}) \neq 0$, this latter complex is never contractible.

If $M$ is closed then $\overline{\mathrm{H}}^*_{(2)}(M)$ is independent of the Riemannian metric on $M$. This is no longer true if $M$ is not closed – consider $\mathbf{R}^2$ and $H^2$. However, the $L^2$-cohomology groups of $M$ do have some invariance properties which we now discuss.

DEFINITION 2. *Riemannian manifolds $M$ and $M'$ are biLipschitz diffeomorphic if there is a diffeomorphism $F : M \to M'$ and a constant $K > 1$ such that the Riemannian metrics $g$ and $g'$ satisfy the pointwise inequality*

$$K^{-1} g \leq F^* g' \leq K g \,. \tag{2.10}$$

If $M$ and $M'$ are biLipschitz diffeomorphic then their reduced and unreduced $L^2$-cohomology groups are isomorphic, as the Riemannian metric only enters in the complex (2.7) in determining which forms are square-integrable. Thus the answer to the zero-in-the-spectrum question only depends on the biLipschitz diffeomorphism class of $M$. More generally, we can consider a category whose objects are Lipschitz Riemannian manifolds and whose morphisms are Lipschitz maps. Then the reduced and unreduced $L^2$-cohomology groups are Lipschitz-homotopy-invariants.

Note that $L^2$-cohomology groups are not coarse quasi-isometry invariants. For example, any closed manifold is coarsely quasi-isometric to a point, but its $L^2$-cohomology is the same as its ordinary complex-valued cohomology, which may not be that of a point. However, some aspects of $L^2$-cohomology only depend on the large-scale geometry of the manifold.

PROPOSITION 2 ([20], Prop. 12). *If $M$ and $M'$ are isometric outside of compact sets then*

1. $\mathrm{Ker}(\triangle_p)$ *is finite-dimensional on $M$ if and only if it is finite-dimensional on $M'$.*
2. *Zero is in $\sigma\left(\triangle_p \text{ on } \Lambda^p/\mathrm{Ker}(d)\right)$ on $M$ if and only if the same statement is true on $M'$.*

Consider uniformly contractible Riemannian manifolds of bounded geometry. If two such manifolds are coarsely quasi-isometric then they are Lipschitz-homotopy-equivalent and hence their $L^2$-cohomology groups are isomorphic [15, p. 219]. The next proposition gives an extension of this result in which uniform contractibility is replaced by uniform vanishing of cohomology, the latter being defined as follows.

DEFINITION 3. *We say that* $\mathrm{H}^j(M;\mathbf{C})$ *vanishes uniformly if for all* $r>0$, *there is an* $R(r)\geq r$ *such that for all* $m\in M$,

$$\mathrm{Im}\left(H^j\left(B_{R(r)}(m);\mathbf{C}\right)\to H^j\left(B_r(m);\mathbf{C}\right)\right)=0\,. \tag{2.11}$$

PROPOSITION 3 (Pansu [25]). *Consider a Riemannian manifold* $M$ *of bounded geometry such that for some* $k>0$, $\mathrm{H}^j(M;\mathbf{C})$ *vanishes uniformly for* $1\leq j\leq k$. *Then within the class of such manifolds,*

1. $\overline{\mathrm{H}}^p_{(2)}(M)$ *and* $\mathrm{H}^p_{(2)}(M)$ *are coarse quasi-isometry invariants for* $0\leq p\leq k$.
2. $\mathrm{Ker}\left(\overline{\mathrm{H}}^{k+1}_{(2)}(M)\to\mathrm{H}^{k+1}(M;\mathbf{C})\right)$ *and* $\mathrm{Ker}\left(\mathrm{H}^{k+1}_{(2)}(M)\to\mathrm{H}^{k+1}(M;\mathbf{C})\right)$ *are coarse quasi-isometry invariants.*

## 3. GENERAL PROPERTIES OF $L^2$-COHOMOLOGY

In this section we give some general results about the $L^2$-cohomology of complete Riemannian manifolds. First, we give a useful sufficient condition for the reduced $L^2$-cohomology to be nonzero.

PROPOSITION 4. *For all* $p$, $\mathrm{Im}\left(\mathrm{H}^p_c(M;\mathbf{C})\to\mathrm{H}^p(M;\mathbf{C})\right)$ *injects into* $\overline{\mathrm{H}}^p_{(2)}(M)$.

*Proof.* Suppose that $\omega$ is a smooth compactly-supported closed $p$-form which represents a nonzero class in $\mathrm{H}^p(M;\mathbf{C})$. By Poincaré duality, there is a smooth compactly-supported closed $\left(\dim(M)-p\right)$-form $\rho$ such that $\int_M\omega\wedge\rho\neq 0$.

As $\omega$ is compactly-supported, it is square-integrable and so represents an element $[\omega]$ of $\overline{\mathrm{H}}^p_{(2)}(M)$. Suppose that $[\omega]=0$. Then there is a sequence $\{\eta_i\}_{i=1}^{\infty}$ in $\Omega^{p-1}(M)$ such that $\omega=\lim_{i\to\infty}d\eta_i$, where the limit is in an $L^2$-sense. It follows that

$$\int_M\omega\wedge\rho=\lim_{i\to\infty}\int_M d\eta_i\wedge\rho=\lim_{i\to\infty}\int_M d(\eta_i\wedge\rho)=0\,, \tag{3.1}$$

which is a contradiction. Thus $[\omega]\neq 0$. □