3. Rings of integers

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **42 (1996)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **09.08.2024**

(2.5) PROPOSITION. *If N and M are factor equivalent then for any $A[G]$-linear embedding $j\colon M \rightarrowtail N$ the function $H \mapsto \left[N^H : j(M^H)\right]_A$ is factorizable.*

Proof. We have $j = \varphi i$, where i is an embedding as in (2.4) and φ is a $K[G]$-linear automorphism of $N \otimes_A K$. Using [15, Ch. III, § 1, Prop. 2] and the notation of (2.3) we see that

$$\left[N^H : j(M^H)\right]_A = d_\varphi(H) \cdot \left[N^H : i(M^H)\right]_A \ .$$

This is a product of two factorizable functions by (2.3) and by our choice of i. □

The fact that "factor equivalence" is an equivalence relation is an easy consequence of (2.5). If $\mathfrak{p}$ is a prime of K not dividing $\#G$ then condition (1) of (2.4) implies that the $\mathfrak{p}$-part of $\left[N^H : i(M)^H\right]_A$ is factorizable. One can prove this with [16, § 15.2] and [16, § 14.4, Lemma 21].

(2.6) REMARK. The definitions of factorizability given by Fröhlich [8; 9] and Burns [2] for abelian groups G are in agreement with our definitions. They also define the notion called $\mathbf{Q}$-factorizability in the abelian case, which is a stronger condition than factorizability. However, the function that one wants to be factorizable in the definition of factor equivalence automatically satisfies this stronger condition if it is factorizable. Thus, $\mathbf{Q}$-factor equivalence is the same as factor equivalence.

In [4, § 3] a factorizable function f with values in $I(\mathbf{Q})$ must also satisfy an additional condition : there should be a map g from the group of complex characters $R_{\mathbf{C}}(G)$ to $I(E)$, where E is some normal number field containing all character values of G, such that g is $\mathrm{Gal}(E/\mathbf{Q})$-equivariant, and such that $g(1_H^G)$ is the E-ideal generated by $f(H)$. It is not hard to see that this condition is satisfied by all functions that are factorizable in our sense.

3. RINGS OF INTEGERS

Let A be a Dedekind domain with quotient field K of characteristic zero and let L a Galois extension of K with Galois group G. The integral closure B of A in L is again a Dedekind domain. Assume that for all primes of L the residue class field extension is separable.

(3.1) THEOREM. *The $A[G]$-lattices B and $A[G]$ are factor equivalent.*

Proof. Define a $B[G]$-module structure on $B \otimes_A B$ by letting B act on the left factor and G on the right. We will show first that $B \otimes_A B$ and $B[G]$ are factor equivalent as $B[G]$-lattices. Define the canonical $B[G]$-linear map $\varphi\colon B \otimes_A B \to B[G]$ by

$$x \otimes y \mapsto \sum_{\sigma \in G} x\sigma(y) \cdot \sigma^{-1} .$$

Let H be a subgroup of G. If $\sigma_1, \dots, \sigma_n$ are the K-embeddings of L^H in L, and if there is an A-basis $\omega_1, \dots, \omega_n$ of B^H, then the restriction $(B \otimes_A B)^H \to B[G]^H$ of φ is a B-linear map with matrix $\big(\sigma_i(\omega_j)\big)_{ij}$ on the bases $\{1 \otimes \omega_j\}$ and $\{b_i\}$, where b_i is the formal sum of those $\sigma \in G$ for which σ^{-1} restricts to σ_i. The square of the determinant of this matrix generates the discriminant $\Delta(B^H/A)$ as an A-ideal. By localization it follows that even if B is not free over A, we have

$$\left[B[G]^H : \varphi(B \otimes_A B)^H\right]_B^2 = \Delta(B^H/A) \cdot B .$$

By Hasse's conductor discriminant product formula [15, Ch. VI, §3] the ideal $\Delta(B^H/A)$ is a factorizable function of H, so $B \otimes_A B$ and $B[G]$ are factor equivalent $B[G]$-lattices.

In order to descend to $A[G]$-lattices, note that there exists an $A[G]$-linear injection $i\colon A[G] \to B$ by the normal basis theorem, and consider the induced $B[G]$-linear map $i_*\colon B[G] \to B \otimes_A B$ that sends $b\sigma$ to $b \otimes i(\sigma)$ for $b \in B$ and $\sigma \in G$. We have

$$\left[(B \otimes_A B)^H : i_*\big(B[G]\big)^H\right]_B = \left[B^H : i\big(A[G]\big)^H\right]_A \cdot B ,$$

and by (2.5) we know that the left hand side is a factorizable function of H. But then the A-index $\left[B^H : i\big(A[G]\big)^H\right]_A$ is also factorizable. □

4. S-UNITS

Let L/K be a Galois extension of number fields with Galois group G, and let S be a finite G-stable set of primes of L containing the infinite primes. The ring of S-integers of L consists of all elements of L that are integral outside S. Its class number is written as $h_S(L)$ and its unit group, the group of S-units of L, is denoted by $U_S(L)$. The group of roots of unity in L is denoted by μ_L and its order is written as $w(L)$.