Introduction

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DOUBLE VALUED REFLECTION IN THE COMPLEX PLANE

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INTRODUCTION

Single valued global reflection in straight lines and circles in the complex plane, introduced into function theory and developed mainly by H.A. Schwarz, has been used to great effect. Local reflection in real analytic curves as given by Schwarz and Caratheodory also plays a very important role. To many such curves one can associate global reflections, which are however multiple valued. While Caratheodory, for example, was certainly aware of this for the ellipse, the systematic theory of multiple valued reflection in one complex dimension seems to have gone undeveloped until now.

In this paper we consider the simplest curves γ in the complex plane **C** which are invariant under a double valued reflection. These include the conics, which are considered in section 2, and certain cubic and quartic curves given in section 1. We then characterize the situation intrinsically in section 3. The data consists of a pair $\{\tau_1, \tau_2\}$ of holomorphic involutions on the complexification Γ of γ, with $\tau_2 = \rho\tau_1\rho$, where ρ is the anti-holomorphic involution of Γ fixing γ. Actually, double valued reflections were first studied in several complex variables, in the context of non-degenerate complex tangents [8]. The importance of studying the dynamics of the *reversible* map $\sigma = \tau_1\tau_2$ was brought out in [8]. Some of the simpler results of [8] are used here to classify involution pairs when Γ is the Riemann sphere. As an application we show how some of the classically known Riemann maps, in somewhat different form, follow systematically from our theory.

We then go on to consider involutions on a one dimensional complex torus in sections 5 and 6. This is mostly classical in nature, although the realization by explicit algebraic equations of the kind needed here does not seem to be in the literature. In the final section we consider a special case

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involving a rectangular lattice. We give the Riemann map for the domain bounded by one branch of the associated real quartic (elliptic) curve in terms of a Weierstrass sigma quotient and the $\mathscr{P}$-function.

1. Double valued reflection

To place our work in context, we first consider, somewhat informally, the general concept of an anti-holomorphic involutive correspondence, or multiple valued reflection, on a complex manifold $\mathscr{U}$. This is an assignment $z \mapsto Q_z$, of a complex subvariety $Q_z \subset \mathscr{U}$ to each point $z \in \mathscr{U}$, such that

$$w \in Q_z \Leftrightarrow z \in Q_w \, . \tag{1.1}$$

The variety Q_z depends antiholomorphically on the point z in a way which can be made precise. The "fixed point set" is the set

$$\gamma = \{z \in \mathscr{U} \mid z \in Q_z\} \, . \tag{1.2}$$

Such a correspondence is double valued if each Q_z is generically zero dimensional and contains two points. Starting from a generic point $z_0 \in \mathscr{U}$, we have $Q_{z_0} = \{z_1, z_1'\}$. Choosing z_1, we get $Q_{z_1} = \{z_0, z_2\}$, $Q_{z_2} = \{z_1, z_3\}, \ldots$ Thus we generate a sequence

$$z_0 \mapsto z_1 \mapsto z_2 \mapsto z_3 \cdots , \tag{1.3}$$

with z_{2k} and z_{2k+1} locally determined and depending holomorphically, respectively, antiholomorphically, on z_0. If we choose z_1', then $Q_{z_1'} = \{z_0, z_2'\}$, $Q_{z_2'} = \{z_1', z_3'\}, \ldots$, and we generate a similar sequence

$$z_0 \mapsto z_1' \mapsto z_2' \mapsto z_3' \cdots . \tag{1.4}$$

A basic problem of the theory is to understand the dynamics of this process. We shall make the foregoing more precise, but only in the case where $\mathscr{U}$ is an open subset of the complex plane.

Let $r(z, \zeta)$ be holomorphic on $\mathscr{U} \times \bar{\mathscr{U}}$ where

$$\mathscr{U} \subseteq \mathbf{C}, \ \bar{\mathscr{U}} = \{\bar{z} \mid z \in \mathscr{U}\} , \tag{1.5}$$

which satisfies

$$r \circ \rho = \bar{r}, \ \rho(z, \zeta) = (\bar{\zeta}, \bar{z}) \, . \tag{1.6}$$