2. Conic sections

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We make $a_2 = 1$ and then restrict to $c = \bar{c}$, $\lambda c^3 = 1$. If $b_0 \neq 0$, we can make $b_0 = 1$, which gives the normal form, under the Moebius group,

$$r = 1 + bz\bar{z} + 2Re(az + z^2\bar{z}),\ b \in \mathbf{R},\ a \in \mathbf{C}\ . \tag{1.17}$$

If $b_0 = 0$, we have

$$a_0 \mapsto \lambda c a_0 = c^{-2} a_0\ .$$

We make $|a_0| = 1$, after which we must restrict to $c = \pm 1$, $\lambda = \mp 1$. The normal form is

$$r = bz\bar{z} + 2Re(az + z^2\bar{z}),\ |a| = 1,\ b \geqslant 0\ . \tag{1.18}$$

In summary we have proved the following.

PROPOSITION 1.1. *Suppose that the (non-empty) real algebraic curve* $\gamma \subset \mathbf{C}$ *admits double valued reflection. Then, under Moebius transformation* γ *is equivalent to a conic section, or to a curve* $r = 0$, *where* r *is given by either (1.17) or (1.18).*

A different normal form will appear later from the intrinsic point of view.

We note that if $\deg_z r = 1$, then we have a circle, and the above process reduces to transforming it to a straight line. These are the cases of global single valued reflection.

2. CONIC SECTIONS

In this section we shall describe the relevant geometry of real quadratic curves in the complex plane. This should give a clearer idea of the possible dynamics in (1.3) and (1.4). The description is only "local" in that it depends on making certain branch cuts, so that the double valued reflection falls into two single valued reflections. In the next section we shall give a more coherent treatment, essentially by passing to a two-sheeted Riemann surface, namely Γ, on which the double valued reflection becomes a single valued one, namely ρ.

The conic with foci $\pm a$, $a > 0$, and parameter $b > 0$ is given by

$$|z + a| + \varepsilon |z - a| = b,\ \varepsilon^2 = 1\ . \tag{2.1}$$

This is an ellipse if $\varepsilon = +1$, $b > 2a$, and one branch of a hyperbola if $\varepsilon = -1$, $b < 2a$. The other branch is gotten by replacing b with $-b$. Squaring and simplifying twice gives the equation

$$r(z, \bar{z}) \equiv Bz\bar{z} - A(z^2 + \bar{z}^2) - 1 = 0 , \tag{2.2}$$

where

$$A = \frac{4a^2}{b^2(b^2 - 4a^2)} , \quad B = \frac{4(b^2 - 2a^2)}{b^2(b^2 - 4a^2)} , \tag{2.3}$$

$$a^2 = \frac{4A}{B^2 - 4A^2} , \quad b^2 = \frac{4}{B - 2A} . \tag{2.4}$$

(2.3) shows that (2.1) is an ellipse when $A > 0$ and a hyperbola when $A < 0$. (2.4) shows that $B - 2A$ is always positive, and that $A, B + 2A$, and $B^2 - 4A^2$ all have the same sign. Conversely, if (2.2) represents a conic with foci on the x-axis ($z = x + iy$), then we must have $B - 2A > 0$. It is an ellipse when $B + 2A > 0$, hyperbola when $B + 2A < 0$, with vertices at $\pm a_1$,

$$a_1 = \sqrt{B - 2A} . \tag{2.5}$$

The complexified conic Γ is given by ($\zeta = \bar{w}$)

$$r(z, \bar{w}) = Bz\bar{w} - A(z^2 + \bar{w}^2) - 1 = 0 . \tag{2.6}$$

For each fixed z we get two values of $\bar{w}$, except when the discriminant

$$\Delta = (B^2 - 4A^2)z^2 - 4A \tag{2.7}$$

vanishes, i.e. when $z = \pm a$ is a focus. If we cut the z-plane from one focus to the other along a segment on the Riemann sphere $\mathbf{P}_1$, the Riemann surface $\pi_1 : \Gamma \to \mathbf{P}_1$ falls into two sheets and the double valued reflection splits into two single valued ones, which we denote by ρ_+, ρ_-. They are most easily visualized via the sine transform

$$z = a \sin t = a(\sin t' \cosh t'' + i \cos t' \sinh t''), t = t' + it'' . \tag{2.8}$$

For the *ellipse* we make the cut from $+a$ to $+\infty$ along the positive real axis and from $-\infty$ to $-a$ along the negative real axis. The remaining open set is the biholomorphic image of the strip $|t'| < \pi/2$ under (2.8). For $c'' > 0$, the pair of segments $t'' = \pm c''$ are transformed into one confocal ellipse, denoted $E_{c''}$, while for $|c'| < \pi/2$, the vertical line $t' = c'$ is mapped to one branch $H_{c'}$ of a confocal hyperbola. For

$$c_0'' = \cosh^{-1}\left(\frac{a_1}{a}\right) = \cosh^{-1}\sqrt{\frac{B + 2A}{4A}} , \tag{2.9}$$

$E_{c_0''}$ is our original ellipse. It follows that in the t-coordinate the above mentioned reflections $\rho_\pm$ are given by

$$\rho_\pm(t', t'') = (t', \pm 2c_0'' - t'') , \tag{2.10}$$

while the first component of σ is given by

$$\rho_- \circ \rho_+(t', t'') = (t', t'' - 4c_0'') . \tag{2.11}$$

Note that these maps preserve each confocal hyperbola branch, while permuting the confocal ellipses. By successive reflections in $t'' = c_0''$ and $t'' = -c_0''$, a suitable analytic function defined inside the original ellipse can be extended to a larger and larger domain, eventually to the entire cut z-plane.

Note that by reflecting the segment $t'' = c_0''$ in the segment $t'' = -c_0''$, and vice-versa, we see that to each point of $E_{c_0''}$ there is a unique point $z_1(z) \in E_{3c_0''}$ (lying on the same confocal hyperbola), with $r(z, \overline{z_1(z)}) = 0$. These points z_1 sweep out the first "self reflection" of the original ellipse.

For the *hyperbola* we cut the z-plane along the finite segment from $-a$ to $+a$ on the real axis. We set

$$c_0' = \sin^{-1}\left(\frac{a_1}{a}\right) = \sin^{-1}\sqrt{\frac{4A}{B+2A}} . \tag{2.12}$$

Then (2.8) maps the lines $t' = c_0'$, and $t' = \pi - c_0'$, onto $H_{c_0''}$, and the line $t' = \pi/2$ two-to-one onto $[a, +\infty)$. The two local single valued reflections are now given by

$$\rho_+(t) = (2c_0' - t', t''), \rho_-(t) = (2(\pi - c_0') - t', t'') , \tag{2.13}$$

in which we may restrict to $t'' > 0$, and

$$\rho_- \circ \rho_+(t) = (t' + 2\pi - 4c_0', t'') . \tag{2.14}$$

By the 2π-periodicity of the sine function, these maps are single valued on the cut z-plane. They preserve each confocal ellipse and permute the branches of the confocal hyperbolas.

We can easily compute the maps τ_i for conics. If $(z', \bar{w}') = \tau_1(z, \bar{w})$, then $z' = z$, and $r(z, \bar{w}') = 0$, $r(z, \bar{w}) = 0$. Substracting these two equations gives

$$\tau_1(z, \bar{w}) = (z, BA^{-1}z - \bar{w}) . \tag{2.15}$$

Similarly,

$$\tau_2(z, \bar{w}) = (-z + BA^{-1}\bar{w}, \bar{w}) , \tag{2.16}$$

and

(2.17) $$\sigma(z, \bar{w}) = (-z + BA^{-1}\bar{w},\ -BA^{-1}z + (B^2A^{-2} - 1)\bar{w}) .$$

It is only the restrictions of these maps to $r(z, \bar{w}) = 0$ which has intrinsic meaning. We have proved the following.

PROPOSITION 2.1. *a) For the ellipse (2.1),* $\varepsilon = +1$, *let the z-plane be cut along the semi-infinite segments* $(-\infty, -a]$ *and* $[+a, +\infty)$ *of the real axis. Then the z-component of* σ *becomes single valued and preserves each confocal hyperbola branch while permuting the confocal ellipses. It is conjugate, via the sine transform (2.8), to the translation map (2.11) of the infinite strip. b) For the hyperbola,* $\varepsilon = -1$, *let the z-plane be cut along the finite segment* $[-a, +a]$ *of the real axis. The z-component of* σ *becomes single valued and preserves each confocal ellipse while permuting the confocal hyperbola branches. It is conjugate to the map covered by the translation map (2.14) on the upper half plane.*

The proposition demonstrates a certain vague principle first brought out in [8]: *elliptic geometry leads to hyperbolic dynamics,* while *hyperbolic geometry leads to elliptic dynamics.*

Finally, we consider a parabola with focus at $z = 0$, vertex at $z = a/2$, and directrix line $Re\,z = a > 0$,

$$a - Re\,z = |z| .$$

Simplifying as before, we get

(2.18) $$r(z, \bar{w}) \equiv (z - \bar{w})^2 - 4a(z + \bar{w}) + 4a^2 = 0 ,$$

with discriminant $\Delta = 8az$. Proceeding as before we find

(2.19) $$\tau_1(z, \bar{w}) = (z, 2z - \bar{w} + 4a),\ \tau_2(z, \bar{w}) = (-z + 2\bar{w} + 4a, \bar{w}) ,$$

and

(2.20) $$\sigma(z, \bar{w}) = (-z + 2\bar{w} + 4a,\ -2z + 3\bar{w} + 12a) .$$

Again these maps must be restricted to the curve (2.18).

The squaring map $z = t^2$ plays the role analogous to the sine transform above. We may cut the z-plane from 0 to $+\infty$ along the real axis, and consider the reflections in $t' = \sqrt{a/2}$, $t' = -\sqrt{a/2}$, $t'' > 0$. The parabolic nature of the dynamics becomes clear.