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KLYACHKO'S METHODS AND THE SOLUTION OF EQUATIONS OVER TORSION-FREE GROUPS

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SUMMARY. The question we are concerned with here is the following:

Let G be a torsion-free group and consider the free product $G * \langle t \rangle$ of G with an infinite cyclic group (generator t). Let w be an element of $G * \langle t \rangle - G$ and $\langle\langle w \rangle\rangle$ denote the normal closure of w in $G * \langle t \rangle$, then is the natural homomorphism

$$G \to \frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$$

injective?

Klyachko's paper: "Funny property of sphere and equations over groups" [Kl] contains a proof that it is injective in the case in which the exponent sum of t in w is 1. If the exponent sum is not ± 1 then $\frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$ has a non-trivial cyclic quotient. So the following is implied:

COROLLARY (Kervaire conjecture for torsion-free groups). *Let G be a non-trivial torsion-free group then $\frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$ is non-trivial.*

The proof for exponent sum 1 is based on Klyachko's "funny property of sphere". This is the following: Let K be a cell subdivision of the 2-sphere with at least one 1-cell. Let a car drive round the boundary of each 2-cell in an anti-clockwise sense (the cars travel at arbitrary speeds, never stop and visit each point of the boundary of the cell infinitely often). Then there must be at least two places on the sphere where complete crashes occur (a complete crash is either a head-on collision in the middle of a 1-cell or a crash at a vertex involving all the cars from neighbouring 2-cells).

Klyachko describes this property as "suitable for a school mathematics tournament". The property is used to show that the diagram for a potential counterexample to the Kervaire conjecture must have at least one interior vertex with all labels being the same element of G, hence this element has finite order.

In this paper we shall give an exposition of Klyachko's methods and theorems. We use his techniques to give a positive answer to the question for other exponents under a technical condition on the t-shape of w, for details here see section 5.