3. TWO TRANSVERSALITY LEMMAS

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notation. Let r_i be the road common to the boundary of D and the country with stopping car κ_i. Let the end junctions of r_i be v_i and v_{i+1} in anticlockwise order $i = 1, 2, \dots$. Suppose κ_i is on the road r_i. Then by hypothesis the roads r_{i+1} and r_{i-1} are free of the cars κ_{i+1} and κ_{i-1} respectively (see figure 3).

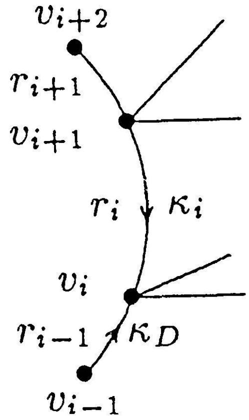

FIGURE 3
The motion of car κ_D

As κ_i traverses r_i from v_{i+1} to v_i let κ_D traverse r_{i-1} from v_{i-1} to v_i. Let the cars meet at v_i at time t. This will not be a complete crash since κ_{i-1} is missing. Again by hypothesis κ_{i+1} will not be at v_{i+2} at time t. Let r be largest such that κ_{i+r} is not at v_{i+r+1} at time t. Then κ_D has enough time to arrive at v_{i+r+1} just as κ_{i+r} does. If there is no such r then let κ_{i+r} be the next car to arrive at D and let κ_D go once round the entire boundary and arrive at v_{i+r+1} just as κ_{i+r} does. Keep repeating this strategem to define the motion of κ_D.

Now we are in a situation corresponding to the first crash theorem and the result is proved. □

3. Two transversality lemmas

In this section we use transversality (cf. [BRS, F]) to prove the existence of diagrams of van-Kampen type for the two situations that we shall meet in the applications to group theory of the crash theorems (in sections 4, 5 and 6). These lemmas need to be stated very carefully and a failure to do so is one of the major weaknesses in Klyachko's version. The lemmas use the idea of a *corner* of a 2-cell in a cell subdivision K of the 2-sphere. This can be regarded as the (oriented) angle formed by the two adjacent edges meeting at a 0-cell in the boundary of the 2-cell. If all the corners of a 2-cell are labelled by elements of a group, then a word can be read around the 2-cell boundary by composing these elements either unchanged or inverted according as the orientation of the corner agrees or disagrees with that of the 2-cell boundary. Similarly if all the corners at a 0-cell are labelled then a word can be read around that 0-cell. We shall always orient corners *clockwise,* thus if the above words are read *clockwise* for 0-cells and *anticlockwise* for 2-cells, then no inversion is necessary (see figure 4).

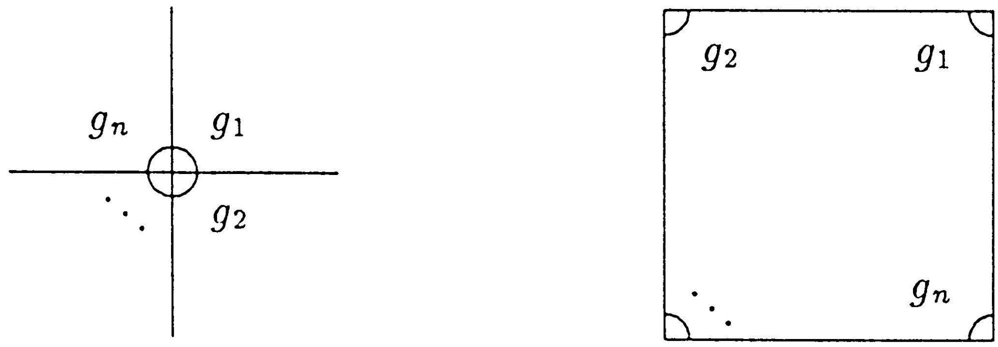

FIGURE 4
Multiplying the corner labels to get $g_1 g_2 \cdots g_n$ for a 0-cell and a 2-cell

Let $w \in A * B$ be an element in the free product of two groups. We shall only be interested in w up to cyclic reordering and thus (cyclically reordering if necessary) we can assume that either $w = 1$ or $w \in A \cup B - 1$ or w is written uniquely as $a_1 b_1 a_2 b_2 \cdots a_n b_n$ where a_i and b_i are non-trivial elements of A, B alternately. These non-trivial elements a_i, b_i are then called the (cyclic) *factors* of w.

LEMMA 3.1. *Let A, B be two groups and let $N = \langle\langle W \rangle\rangle$ be the the normal closure in $A * B$ of some subset of elements $W \subset A * B$. Suppose $N \cap A \neq \{1\}$. Then there is a cell subdivision of the sphere S^2 such that each corner of each 2-cell is labelled by an element of $A \cup B$ with the following properties.*

1. *The corner labels of a 2-cell are the cyclic factors (in anticlockwise order and up to cyclic rotation) of some w or w^{-1} where $w \in W$.*
2. *The corner labels at a 0-cell are either all in A or all in B.*
3. *The (clockwise) product of the corner labels at a 0-cell is 1 (in A or B) except for one special 0-cell where the product is a non-trivial element of $A \cup B$.*

Proof. Let K_A, K_B be two disjoint 2-dimensional complexes such that $\pi_1(K_A, *_A) = A$ and $\pi_1(K_B, *_B) = B$. Join the base points $*_A$ and $*_B$ by an arc α with central point $*$. Let $K = K_A \cup \alpha \cup K_B$. Then $\pi_1(K, *)$

$\cong A * B$. Attach 2-cells σ_w to K by the words $w \in W$ to form the complex L. If $a \in N \cap A - \{1\}$ there is a map $f: D^2, S^1 \to L, K$ from the 2-disc to L such that the restriction $f \mid S^1$ to the boundary represents a. Make the map f transverse to the centres of the 2-cells σ_w. It follows that the inverse images of small neighbourhoods of these centres is a collection of disjoint discs $D_1, \ldots, D_m$ in the interior of D^2. By a radial expansion of f on these discs we may assume that each image is the whole of one of the σ_w. It follows that the punctured disc $P = D^2 - \overline{D_1 \cup \cdots \cup D_m}$ is mapped by f to K. Make $f \mid P$ transverse to $*$. Then $f^{-1}*$ is a 1-manifold Z properly embedded in P. By a radial expansion along α we can assume that Z has a neighbourhood N which is a normal I-bundle and where each fibre is mapped by f to α. The complementary space $P - N$ is divided into connected regions which are mapped by f to K_A or K_B. On crossing N one passes from one kind of region to the other.

We now simplify the subset $D_1 \cup \cdots \cup D_m \cup N$ of D^2 as follows. Suppose N contains an annulus component $\mathscr{A}$ in the interior of P. Let D' denote the interior disc of D^2 which bounds the interior boundary component of the annulus. Then $D' \cup \mathscr{A}$ is a sub disc of D^2 whose boundary gets mapped to a base point by f. We can then shrink it to a point, redefine f and simplify the situation. Having eliminated all annuli, $D_1 \cup \cdots \cup D_m \cup N$ will look like a thickened graph in D^2 with the discs D_i corresponding to thickened vertices and the components of N to thickened edges. Our next task is to make this graph connected. If not choose an innermost component C. Draw a simple loop around C separating it from the rest of $D_1 \cup \cdots \cup D_m \cup N$. This loop will represent (up to conjugacy) an element of $A \cup B$. If this element is trivial we can shrink the disc it bounds as above and simplify the situation. If not we replace $D_1 \cup \cdots \cup D_m \cup N$ by C. Note that the boundary curve may now represent a non trivial element of B instead of A.

Attach a 2-cell (outside) to the boundary of D^2 and label the centre of this outside cell ∞. The 2-disc has now become a 2-sphere. In this situation consider the dual graph Γ. This has a vertex in each region and an edge joining neighbouring regions separated by a component of N. For the outer region take the vertex to be ∞. Then Γ and its complementary regions define a cell subdivision K of the 2-sphere. Each vertex is either in an A region or a B region and the corners can be correspondingly labelled by elements of A or B as follows. Every 2-cell of K contains a unique subdisc D_i. Opposite a corner is an edge of D_i labelled by an element of A or B. Take this to be the labelling of the corner. By moving anticlockwise around the boundary of a 2-cell of K the corner labellings spell out a cyclic rotation of some w_i or w_i^{-1}. By moving clockwise around a 0-cell of K the corner labellings spell out the trivial element (of A or B) except for ∞ which spells out a non-trivial element of A or B. □

NOTE. It may not be possible to specify that the non-trivial element lies in A as this simple example shows. Let $A = \langle a \rangle$, $B = \langle b \rangle$ be two infinite cyclic groups generated by a, b respectively. Let the words of the attaching 2-cells be ab^{-1}, b. In this case the 2-cells of the required subdivision have either two corners (those modelled on ab^{-1}) or one corner (modelled on b) and the only possible subdivision of the 2-sphere satisfying lemma 3.1 is the trivial one with single vertex labelled b. This is a place where Klyachko's version is definitely wrong (rather than badly stated).

Let $w \in G * \langle t \rangle$ be an element of the free product of a group G with the infinite cyclic group $\langle t \rangle$. Then w can be written uniquely (up to cyclic rotation) in the form $w = g_1 t^{\varepsilon_1} g_2 \cdots t^{\varepsilon_n}$ where each $g_i \in G$, each $\varepsilon_i = \pm 1$ and g_i can only be 1 if it has neighbouring t's (in cyclic order) with the same exponent. We call $g_1, \ldots, g_n$ the *coefficients* of w.

The following lemma is proved in [H$_1$]. It is closely related to "pictures" [R$_1$, R$_2$, Sh].

LEMMA 3.2. *Let G be a group and consider the free product $G * \langle t \rangle$ of G with an infinite cyclic group (generator t). Let $N = \langle\langle W \rangle\rangle$ be the the normal closure in $G * \langle t \rangle$ of some subset of elements $W \subset G * \langle t \rangle$. Suppose $N \cap G \neq \{1\}$ then there is a cell subdivision K of the 2-sphere such that*

a) the 1-cells of K are oriented,

b) the corners (all oriented clockwise) are labelled by coefficients of elements of W,

c) the clockwise product of the corner labelling around any 0-cell is 1 except for one vertex where it is non trivial,

d) the corner labels of any 2-cell (in anticlockwise order) are the coefficients of w or w^{-1} for some $w \in W$ (up to cyclic rotation) with the property that, if on passing from one corner to an adjacent corner the element t or t^{-1} is inserted according to whether the intervening edge is oriented in the same or opposite direction, then the whole of w or w^{-1} is recovered.

Proof. The proof is very similar to 3.1. Let K_G be a 2-dimensional complex such that $\pi_1(K_G, *_G) = G$. Adjoin an oriented 1-cell γ to the base point $*_G$ to form a 2-dimensional complex $K = K_G \vee S^1$ with $\pi_1 K = G * \langle t \rangle$. Attach 2-cells to K by the words $w \in W$ to form L. Since $N \cap G \neq \{1\}$ there is a non contractable loop in K_G represented by a map $f: S^1, 1 \to K_G, *_G$ which can be extended to a map $f: D^2 \to L$.

We now proceed as in the proof of lemma 3.1 with the rôle of $*$ played by a point p in the interior of γ. We construct a graph whose (thickened) vertices are the inverse image of the 2-cells and whose edges are the inverse image of p. By making similar simplifications and passing to an innermost component, as before, we may assume that this graph is connected. Replace D^2 by a sphere as before. The dual subdivision now defines K. The orientation of the 1-cells is determined by the orientation of γ and it only remains to observe that these oriented edges correspond to the new generator t. □

4. Application to the Kervaire problem

In this section we give Klyachko's application of the crash theorems to prove theorem 1.1 in the case in which exponent sum of t in the word w is 1. As remarked in the introduction this implies the Kervaire conjecture for torsion-free groups.

We say that a system of equations $\{w(t) = 1 \mid w \in W\}$ in the variable t, with coefficients in a group G, has a *solution over* G if there is a group $\tilde{G}$ containing G as a subgroup and an element $x \in \tilde{G}$ such that the relations $\{w(x) = 1 \mid w \in W\}$ are satisfied in $\tilde{G}$. It is clear that this is equivalent to the natural map

$$G \to \frac{G * \langle t \rangle}{\langle\langle W \rangle\rangle}$$

being injective, where $\langle\langle W \rangle\rangle$ denotes the normal closure of W in $G * \langle t \rangle$.

Now let H be a subgroup of G and let $g \in G$. We say that g is *free relative* to H if the subgroup $\langle g, H \rangle$ of G generated by g and H is naturally the free product $\langle g \rangle * H$ of an infinite cyclic group $\langle g \rangle$ with H.

We shall apply the crash theorem with stops to prove theorem 4.1 (below) and then use an algebraic trick to deduce the case $\mathrm{ex}(w) = 1$ of theorem 1.1.

If g, h are elements of a group let g^h denote $h^{-1}gh$.