4. Application to the Kervaire problem

Proof. The proof is very similar to 3.1. Let K_G be a 2-dimensional complex such that $\pi_1(K_G, *_G) = G$. Adjoin an oriented 1-cell γ to the base point $*_G$ to form a 2-dimensional complex $K = K_G \vee S^1$ with $\pi_1 K = G * \langle t \rangle$. Attach 2-cells to K by the words $w \in W$ to form L. Since $N \cap G \neq \{1\}$ there is a non contractable loop in K_G represented by a map $f: S^1, 1 \to K_G, *_G$ which can be extended to a map $f: D^2 \to L$.

We now proceed as in the proof of lemma 3.1 with the rôle of $*$ played by a point p in the interior of γ. We construct a graph whose (thickened) vertices are the inverse image of the 2-cells and whose edges are the inverse image of p. By making similar simplifications and passing to an innermost component, as before, we may assume that this graph is connected. Replace D^2 by a sphere as before. The dual subdivision now defines K. The orientation of the 1-cells is determined by the orientation of γ and it only remains to observe that these oriented edges correspond to the new generator t. □

4. Application to the Kervaire problem

In this section we give Klyachko's application of the crash theorems to prove theorem 1.1 in the case in which exponent sum of t in the word w is 1. As remarked in the introduction this implies the Kervaire conjecture for torsion-free groups.

We say that a system of equations $\{w(t) = 1 \mid w \in W\}$ in the variable t, with coefficients in a group G, has a *solution over* G if there is a group $\tilde{G}$ containing G as a subgroup and an element $x \in \tilde{G}$ such that the relations $\{w(x) = 1 \mid w \in W\}$ are satisfied in $\tilde{G}$. It is clear that this is equivalent to the natural map

$$G \to \frac{G * \langle t \rangle}{\langle\langle W \rangle\rangle}$$

being injective, where $\langle\langle W \rangle\rangle$ denotes the normal closure of W in $G * \langle t \rangle$.

Now let H be a subgroup of G and let $g \in G$. We say that g is *free relative* to H if the subgroup $\langle g, H \rangle$ of G generated by g and H is naturally the free product $\langle g \rangle * H$ of an infinite cyclic group $\langle g \rangle$ with H.

We shall apply the crash theorem with stops to prove theorem 4.1 (below) and then use an algebraic trick to deduce the case $\mathrm{ex}(w) = 1$ of theorem 1.1.

If g, h are elements of a group let g^h denote $h^{-1}gh$.

THEOREM 4.1. *Let H and H' be two isomorphic subgroups of a group Γ under the isomorphism $h \to h^{\phi}, h \in H$. Suppose that for each i, a_i, b_i are elements of Γ such that a_i is free relative to H and b_i is free relative to H'. Let c be an arbitrary element of Γ. Then the system of equations*

(1) $$(b_0 a_0^t b_1 a_1^t b_2 a_2^t \cdots b_r a_r^t) ct = 1$$

(2) $$h^{\phi} = h^t, h \in H$$

has a solution over Γ.

Proof. Assume not. Then by the second transversality lemma there is a cell subdivision of the 2-sphere such that, the 1-cells of K are oriented, the 2-cells are of the four types I, I', II and II' illustrated in figure 5 with the corners labelled by elements of G as shown and such that the clockwise product of the corner labelling around any 0-cell is 1 except for one vertex (where it is non-trivial). Assume that K is minimal with these properties.

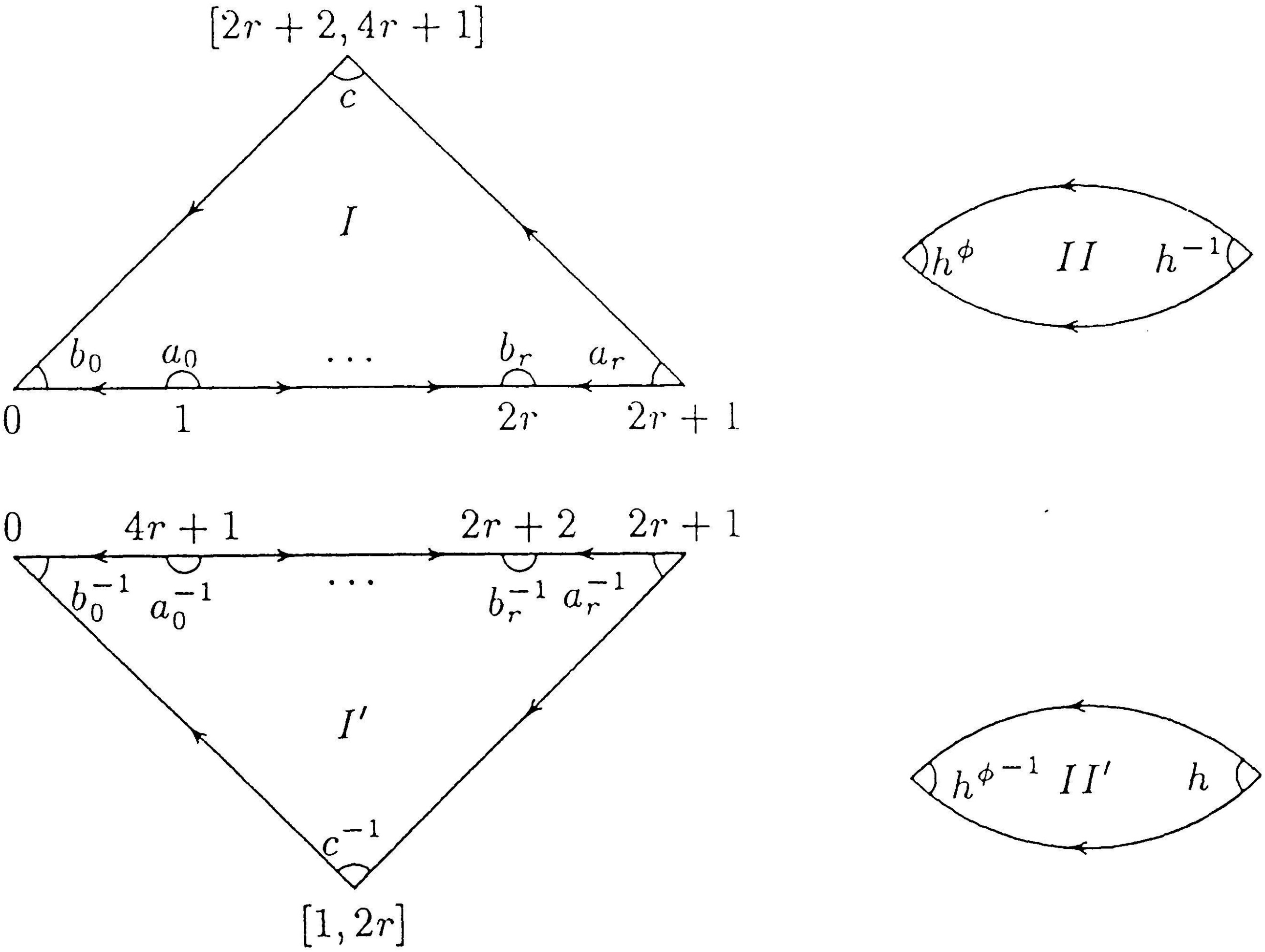

FIGURE 5
The 2-cells I, I', II and II'

A traffic flow is now defined on K as follows. At time 0 let a car on the boundary of a country of type I or I' start at the corner labelled b_0 or b_0^{-1} and proceed in an anticlockwise manner with respect to the orientation of the edge along which it is travelling, moving from corner to corner in unit time except at the corner labelled c or c^{-1} where it stops for $2r - 1$ units. The times when the car is at each corner are illustrated in figure 5. For countries of type II or II' the car starts at the corner labelled h^{ϕ} or $h^{\phi^{-1}}$ and proceeds in an anticlockwise manner moving from corner to corner in unit time.

The fact that the edges are oriented will be used to derive various properties of this flow and of the cell subdivision K. We shall think of the orientation arrows as pointing uphill so that corners come in four types: *top corners,* of which the corner labelled b_0 in 2-cells of type I (see the figure) is an example; *bottom corners*, for example the corner labelled a_0; *up corners*, for example c, and *down corners*, for example c^{-1}. Similarly the 0-cells of K come in three types: *maxima* or sinks, where all the corners are top corners, *minima* or sources, where all the corners are bottom corners and *saddles*, where some of the corners are uphill or downhill. Notice that at a saddle the up and down corners are equal in number and must alternate around the 0-cell (figure 6) although they may be separated by top or bottom corners.

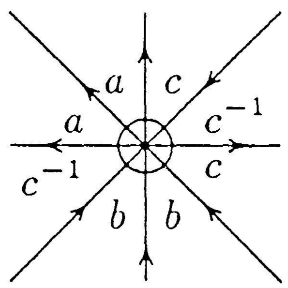

FIGURE 6
Up and down corners alternate around a 0-cell

PROPERTY 1: ONE-WAY FLOW. *If two cars are on roads at the same time then they are both either moving uphill or both downhill.*

Indeed, in intervals of time of the form $(2n, 2n + 1)$, no car is moving uphill and and in intervals of the form $(2n - 1, 2n)$ no car is moving downhill.

Property 1 implies that there are no road crashes.

PROPERTY 2: STOPPING SCHEDULE. *Cars always stop at up and down corners but never at top or bottom corners.*

Property 2 implies that saddles are stopping vertices whilst maxima and minima are not.

PROPERTY 3: SEPARATED STOPS. *It never happens that there is one car at an up corner and at the same time another car at a down corner.*

Property 3 (together with the observation that up and down corners alternate around a saddle) implies that stops are separated, as required for the crash theorem with stops. It also implies that crashes can only occur at maxima or minima.

PROPERTY 4: COHERENCE. *If two cars in cells of type I or I' are both at a non-stop corner, then the corners carry the same label (possibly inverted).*

Thus if a crash occurs at a vertex with labelling giving a trivial product which is a minimum (resp. maximum) then this implies a relationship in $\langle a_i, H\rangle$, (resp. $\langle b_i, H'\rangle$) or that some power of a_i, b_i is trivial for some i. By the hypotheses of the theorem, this can only happen if there is a pair of adjacent corners labelled by a_i, a_i^{-1} or $h, h^{-1}, \ldots$. So there is a neighbouring pair of 2-cells of type I, I' or II, II' which can be removed, simplifying K. It follows that there can only be a total crash at the vertex with non-trivial labelling contradicting the crash theorem with stops which states that there are at least two. □

REMARK. By taking H and H' to be trivial in theorem 4.1, we can now deduce a special case of theorem 1.1, namely the case when the t-shape of w is $t^{-1}tt^{-1}\ldots tt^{-1}tt$. The rest of the section introduces an algebraic trick which will enable us to reduce the general case $\mathrm{ex}(w) = 1$ to this special case.

Let G be a group and consider the homomorphism $\mathrm{ex}\colon G * \langle t\rangle \to \mathbf{Z}$. It is clear that K, the kernel of ex, is generated by elements of the form $g^{t^{\ell}} = t^{-\ell}gt^{\ell}, g \in G$.

Any element of K has a canonical expression of the form $k = g_1^{t^{\ell_1}} \cdots g_r^{t^{\ell_r}}$, where $\ell_i \neq \ell_{i+1}$ for each i. We shall call the $g_i^{t^{\ell_i}}$ the *canonical elements* of k. Let $\min(k)$ be the minimum value of $\ell_i, i = 1, \ldots, r$ and $\max(k)$ the maximum value. Fix a positive integer m. Consider the following subgroups of K:

$$H = \langle k \in K \mid \min(k) \geqslant 0, \max(k) \leqslant m - 2\rangle$$
$$H' = \langle k \in K \mid \min(k) \geqslant 1, \max(k) \leqslant m - 1\rangle$$
$$J = \langle k \in K \mid \min(k) \geqslant 0, \max(k) \leqslant m - 1\rangle$$

and the following subsets:

$$X = \{k \in K \mid \min(k) = 0, \max(k) \leqslant m - 1\}$$
$$Y = \{k \in K \mid \min(k) \geqslant 0, \max(k) = m - 1\}$$
$$Z = \{k \in K \mid \min(k) \geqslant 1, \max(k) = m\} .$$

LEMMA 4.2. *Let* $w \in G * \langle t\rangle$ *satisfy* $\mathrm{ex}(w) = 1$. *Then, after conjugation,* w *can be written as a product*

$$b_0 a_0^t b_1 a_1^t \cdots b_r a_r^t ct ,$$

where $a_i \in Y, b_i \in X, i = 0, \ldots, r$ *and* $c \in J$ *for some* m.

Proof. Clearly w can be written as a product kt where $k \in K$. If $k = 1$ the result is trivial. Otherwise let $k = \prod_1^N g_i^{t^{\ell_i}}$ be the canonical expression for k. Conjugating by a suitable power of t, we can assume that $\min(k) = 0$. Let $m = \max(k)$. Since successive ℓ_i differ, we must have $m \geqslant 1$. Now consider the appearances of the maximum m and the minimum 0 in the canonical expression. Suppose that the first appearance is a minimum. Then by collecting terms up to (but not including) the first maximum, we define an element of X which forms the left part of the canonical expression for k. If the first appearance is a maximum, then we would find an element of Z instead. Continuing in this way we can write k as a product of elements taken alternately from X and Z.

If the first element is from Z, i.e. $k = zxu$ where $z \in Z$, $x \in X$ and u is the rest of the canonical expression, then we conjugate w by z to yield $k't$ where $k' = xuz^{t^{-1}}$. Now the canonical expression for k' may be simpler than that for k and the max may have dropped by 1, but the min is still 0 and now the expression of k' as a product of elements taken alternately from X and Z starts with an element of X.

Thus we may assume that k can be written as a product

$$x_0 z_0 x_1 z_1 \ldots x_r z_r c$$

where $x_i \in X$, $z_i \in Z$ and $c = 1$ or $c \in X$ (and notice that $c \in J$ in either case). Now let $b_i = x_i$ and $a_i^t = z_i$ then $w = kt$ has the required expression. $\square$

LEMMA 4.3. *Suppose that* G *is torsion free then any element* a *of* Y *is free relative to* H. *Similary any element* b *of* X *is free relative to* H'.

Proof. Suppose that a lies in Y. The case $b \in X$ is similar. Let $a = h_1 x_1 h_2 x_2 \cdots h_r$ where $x_i = t^{-m+1} g_i t^{m-1}$ and $h_i \in H$. We can assume that $g_1, \ldots, g_{r-1}$ and $h_2, \ldots, h_{r-1}$ are never the identity element. Assume there is a non trivial relationship $w(a, H) = 1$ which is minimal with respect to the number of occurences of a and its inverse a^{-1}. In particular no cancelling pairs aa^{-1} or $a^{-1}a$ can occur in w. Since G is torsion free the reduction of w to 1 can only occur as a sequence of elimination of pairs hh^{-1} where $h \in H$ or xx^{-1}, where x is an x_i or x_j^{-1}.

Now let $a' = x_1 h_2 x_2 \cdots x_{r-1}$ and define the *core* q of a by $a' = pqp^{-1}$ where p has maximal length. Then, in any subword of w of the form $ahah' \cdots$ where a occurs n times, at least every copy of q, together with the p in the first a and the p^{-1} in the last a, must survive after cancellation. Moreover in any subword of the form aha^{-1} since $h \neq 1$ cancellation can only involve combining h_r, h and h_r^{-1} after which no further cancellation is possible. The result is now clear. □

THEOREM 4.4 (Case ex = 1 of theorem 1.1). *Let* G *be a torsion-free group and let* w *be an element of* $G * \langle t \rangle$ *with* $\mathrm{ex}(w) = 1$ *then the equation* $w = 1$ *has a solution over* G.

Proof. By lemma 4.2 we can assume that $w = b_0 a_0^t b_1 a_1^t \cdots b_r a_r^t c t$, where $a_i \in Y$, $b_i \in X$, $i = 0, \ldots, r$ and $c \in J$. We need to think of each a_i, b_i, c as functions of t and for clarity we shall introduce a new variable s. To be precise let

$$w(s, t) \equiv b_0(t) a_0^s(t) \cdots b_r(t) a_r^s(t) c(t) s$$

where s and t are independent variables.

Write Γ for $G * \langle t \rangle$ and let H, H' be the subgroups defined above. There is an isomorphism $\phi: H \to H'$ given by $h^\phi = h^t, h \in H$.

Lemma 4.3 gives the hypotheses of theorem 4.1, which implies that Γ embeds in $\tilde{\Gamma} = \langle \Gamma, s \mid w(s, t) = 1,\ h^s = h^\phi,\ h \in H \rangle$. Now each of the canonical elements of $a_i(t), b_i(t), c(t)$ is either in G or lies in H^t; moreover in $\tilde{\Gamma}$ we have $h^s = h^\phi = h^t$ for each $h \in H$. It follows that $w(s, s) = 1$ in $\tilde{\Gamma}$.

Therefore there is a commuting diagram

$$\begin{array}{ccc} \Gamma & \subset & \tilde{\Gamma} \\ \cup & & \uparrow \\ G & \to & \hat{G} \end{array}$$

Where $\hat{G} = \frac{G * \langle s \rangle}{\langle\langle w \rangle\rangle}$. Thus $G \to \hat{G}$ is injective. □

REMARK. The alert reader will have noticed that the hypotheses of theorem 4.4 can be weakened. All that has been used is that the coefficients of w are of infinite order. Indeed a careful examination of the proof yields the following sharper statement. If $G \to \frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$ is not injective then one of the *separating* coefficients in w has finite order (separating means between a t and a t^{-1}).

5. THE GENERAL CASE

In this section we consider the adjunction problem as stated in the introduction, in its full generality. We continue to work with torsion-free groups. We shall introduce a class of words with exponent not necessarily 1 for which the methods of the previous sections can be adapted to provide a solution to the adjunction problem. We call such words *amenable*. Before defining amenability in general we shall consider a class of simpler words, on which the general definition will be based, these we call *suitable* words.

t-SHAPES, t-SEQUENCES AND SUITABILITY

Consider finite sequences whose elements are t or t^{-1}. We call such a sequence a *t-sequence*. If m is a positive integer let t^m denote the sequence $t, t, \ldots, t$, m times and let t^{-m} denote the sequence $t^{-1}, t^{-1}, \ldots, t^{-1}$, m times. A *clump* is a maximal connected subsequence of the form t^m or t^{-m} where $m > 1$ and these are said to have *order m* and $-m$ respectively. We call a clump of positive order an *up* clump and a clump of negative order a *down* clump. A sequence is *suitable* if it has exactly one up clump which is not the whole sequence and possibly some down clumps, or if it has exactly one down clump which is not the whole sequence and possibly some up clumps.

It follows that, after a possible cyclic rotation or change $t \mapsto t^{-1}$, a suitable sequence has the form

$$t^s t^{-r_0} t t^{-r_1} t \ldots t t^{-r_k}$$