

## 6. Further Applications

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **42 (1996)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **09.08.2024**

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The  $t$ -sequence  $t^n$  is interesting because the adjunction problem is already proved (without the torsion-free hypothesis) for words with this  $t$ -shape [L]. However the methods discussed here do not extend this result to  $t$ -sequences in normal form based on  $t^n$ . An example is  $t^3 t^{-1} t^3 t^{-1}$ .

Another interesting case is the sequence  $tt^{-1}$  which is not amenable. However a simple trick (substitute  $u^2$  for  $t$ ) makes it suitable. Hence theorem 5.1 implies a solution to the adjunction problem (over torsion-free groups) for words of the form  $gtg't^{-1}$ . For words of this shape, torsion-free is a necessary condition as the example in the introduction shows!

We do not yet have a simple test for amenability though it is easy from the definition to write down large classes of amenable sequences. However it can be seen that, speaking very roughly, a sequence is amenable unless it has a uniform slope, like  $t^5 t^{-3} t^5 t^{-3}$  or  $t^3 t^{-3} t^3 t^{-3}$  (slope zero).

### 6. FURTHER APPLICATIONS

We give here the other applications from [Kl] of the crash theorems, not covered above.

**THEOREM 6.1** (Application to free products). *Let  $A, B$  be groups and suppose each (cyclic) factor of  $u \in A * B - A$  has infinite order. Then the natural homomorphism  $A \rightarrow \langle A * B \mid [A, u] = 1 \rangle$  is injective.*

*Proof.* Suppose not. Then the conditions of the first transversality lemma apply and there is a non trivial element  $a \in A$  such that  $a \in \langle\langle [A, u] \rangle\rangle$ . So we have a cell subdivision  $K$  of the 2-sphere such that reading round from the base point  $*$  for every 2-cell in  $K$  spells out the word

$$w(a) = (c_0^{-1} a c_0) c_1 \cdots c_{n-1} (c_n^{-1} a^{-1} c_n) c_{n-1}^{-1} \cdots c_1^{-1}$$

for some  $a \in A$ , see figure 8. Note that if this 2-cell has the opposite orientation then the word spelt out is  $w(a^{-1})$ .

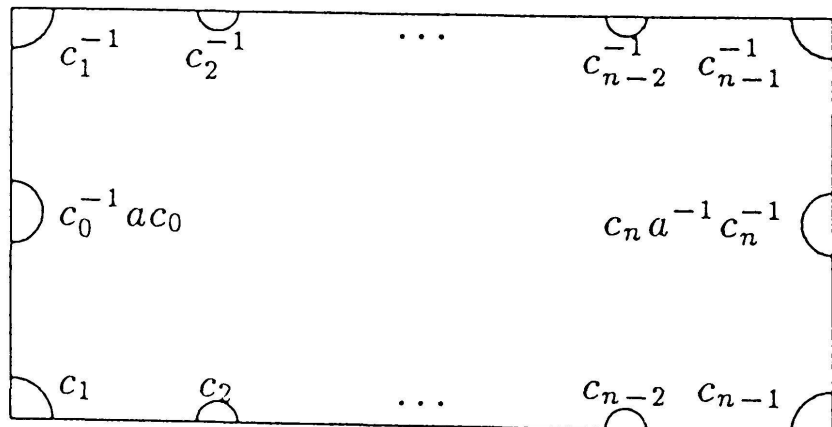


FIGURE 8  
The 2-cell labelled by  $w(a)$

Now consider the traffic flow defined as follows. The car associated to a 2-cell starts out from the base point  $*$  and proceeds in an anticlockwise manner so that it takes a unit amount of time to reach the next corner. It is clear that any crash must take place at a 0-cell. By the crash theorem there are at least two total crashes so we can assume it takes place at a 0-cell where the angle labels multiply to 1. There are two cases to consider.

If the crash occurs at time  $0 \pmod n$  then the clockwise labelling around this 0-cell is  $c_\alpha^{-1} a_1 c_\alpha, c_\alpha^{-1} a_2 c_\alpha, \dots, c_\alpha^{-1} a_k c_\alpha$  where  $a_1 a_2 \cdots a_k = 1$  and  $\alpha = 0$  or  $\alpha = n$ . A simple calculation shows that the anticlockwise product of the remaining angles of these  $k$  2-cells is 1. So we may simplify the situation by collapsing these  $k$  2-cells to a point.

If the crash occurs at a time  $\neq 0 \pmod n$  then the clockwise labelling around this 0-cell is  $c_i^k = 1$  for some  $0 < i < n$  and some  $k > 1$  contradicting the torsion free hypothesis.  $\square$

Let  $H, H'$  be groups and let  $\phi: H \rightarrow H'$  be an isomorphism. We shall use the notation  $h^\phi$  to denote the image of  $h \in H$  under  $\phi$ . Similarly we shall write  $a^b := b^{-1} a b$  for conjugation.

**THEOREM 6.2** (Application to HNN extensions). *Let  $H$  and  $H'$  be two isomorphic subgroups of the group  $A$  under the isomorphism  $h \rightarrow h^\phi, h \in H$ . Let  $B$  be a group and let  $w \in A * B - A$  have torsion free factors. Then the natural map*

$$A \rightarrow \langle A, B \mid w^{-1} h w = h^\phi, h \in H \rangle$$

is injective.

*Proof.* Consider the following groups

$$A' = \langle A, t \mid t^{-1} h t = h^\phi, h \in H \rangle,$$

$$A'' = \langle A, t, B \mid t^{-1} h t = h^\phi, [a, t^{-1} w] = 1, [t, w] = 1, h \in H, a \in A \rangle$$

$$= \langle A', B \mid [a, t^{-1} w] = 1, [t, w] = 1, a \in A \rangle,$$

$$A''' = \langle A, B \mid w^{-1} h w = h^\phi, h \in H \rangle.$$

We can construct the following commuting diagram,

$$\begin{array}{ccccc} & & & & A''' \\ & & & \delta & \\ & & & \nearrow & \downarrow \gamma \\ & & & & A'' \\ A & \xrightarrow{\alpha} & A' & \xrightarrow{\beta} & A'' \end{array}$$

where the maps  $\alpha, \beta, \gamma$  and  $\delta$  are induced by inclusion. In order for  $\gamma$  to be a well defined homomorphism it is necessary to check that the relation  $w^{-1}hw = h^\phi, h \in H$  is a consequence of the relations  $t^{-1}ht = h^\phi, [a, t^{-1}w] = 1, [t, w] = 1, h \in H, a \in A$ . But this follows because  $w^{-1}hw = w^{-1}tt^{-1}htt^{-1}w = w^{-1}th^\phi t^{-1}w = h^\phi$ . Now  $\alpha$  is injective because  $A'$  is an HNN extension of  $A$  (see [DD, p. 33] or [Se, p. 9]) and  $\beta$  is injective because of theorem 6.1. So  $\delta$  is injective and this proves the theorem.  $\square$

THEOREM 6.3. *Let*

$$(*) \quad u_i(t) = 1, i \in I$$

*be a set of equations over the group  $A$  where the exponent sum of  $t$  in each  $u_i(t)$  is zero. Suppose  $w = w(t) \in A * \langle t \rangle - A$  and the factors of  $w$  are all torsion free. Then the set of equations*

$$(**) \quad u_i(w(t)) = 1, i \in I$$

*has a solution over  $A$  if and only if the set (\*) has a solution over  $A$ .*

*Proof.* Let  $w(t) = at$  where  $a \in A$  has infinite order. Then a solution  $x$  for  $u_i(w(t)) = 1$  defines a solution  $at$  for (\*).

Conversely suppose  $x \in A'$  is a solution of the set of equations  $\{u_i(t) = 1 \mid i \in I\}$ . Let  $G$  be the subgroup of  $A'$  generated by

$$\{x^{-n}ax^n \mid a \in A, n \in \mathbf{Z}\}.$$

Then  $A$  is a subgroup of  $G$  and  $G$  is a subgroup of

$$H = \langle G, t \mid w^{-1}gw = g^\phi, g \in G \rangle$$

where  $g^\phi = x^{-1}gx$  by theorem 6.2. Because of the exponent sum condition  $u_i(w) = 1, i \in I$ .  $\square$

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