2. Commutation and stabilisers

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **42 (1996)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **09.08.2024**

Proof. We have already argued that $A * \beta = B$ implies the existence of the ribbon. On the other hand, suppose there is a ribbon R from A to B proper for β. Then by reflection of the ribbon from A to $A * \beta$ and concatenation with R, one has a ribbon from $A * \beta$ to B along $\beta^{-1}\beta$. But $\beta^{-1}\beta$ can be moved by level-preserving isotopy to the trivial braid $\{1, \ldots, n\} \times \mathbf{I}$, and then the image of the ribbon provides an isotopy from $A * \beta$ to B fixing $\{1, \ldots, n\}$. □

2. Commutation and stabilisers

The theme of this paper is to reflect algebraic properties of a braid in the geometry of ribbons and the action of B_n on $\mathbf{A}_n$.

Consider an n-braid β which is constructed from an $(n-1)$-braid by running a narrow ribbon along the j^{th} string, with the ends of the ribbon being straight line segments on the real line, as pictured in Figure 2. The ribbon may be twisted arbitrarily. Let β consist of the two edges of the ribbon, together with the other strands of the $(n-1)$-braid (those of index greater than j need to be renumbered and have their ends shifted, of course.) Premultiplying β by σ_j corresponds to putting a twist in the left end of the ribbon, and the ribbon can be used to convey that twist through β until it emerges on the right, and we have the equation: $\sigma_j\beta = \beta\sigma_k$.

In the special case of $j = k$ we have constructed a class of braids which commutes with the generator σ_j. In fact, if β is any braid for which $[j, j+1] * \beta = [j, j+1]$, it can be isotoped, with fixed endpoints, into one with such parallel strands. Just slide the strands near each other along the ribbon, but taper to the identity to keep the ends fixed.

Definition. We say that β *has a* (j, k)*-band* if there exists a ribbon (the band) proper for β and connecting $[j, j+1] \times 0$ to $[k, k+1] \times 1$.

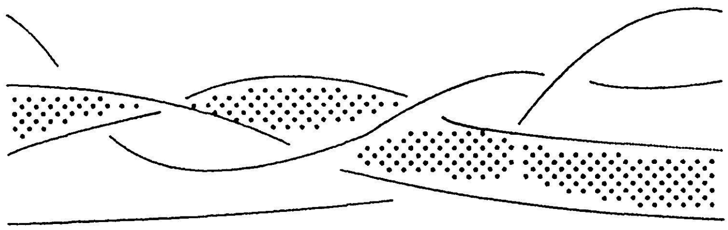

Figure 2
A braid with a (2.1)-band

According to Proposition 1.1, β has a (j, k)-band if and only if $[j, j+1] * \beta = [k, k+1]$. However, it may not be obvious, from an expression as a word in the generators, whether a braid has a (j, k)-band, and subwords of braids with bands may fail to have bands, as illustrated by the following example.

EXAMPLE. Consider the braids $\alpha = \sigma_2^{-1}\sigma_1^2\sigma_2$ and $\beta = \sigma_1\sigma_2^{-2}$. Then $\alpha\beta$ has a (1, 1)-band. But neither α nor β have a (1, 1)-band, although they both stabilise $\{1, 2\}$. The arc $A = [1, 2] * \alpha = \beta * [1, 2]$ is as pictured in Figure 3. It is an interesting exercise for the reader to check that $\alpha\beta$ commutes with σ_1, whereas neither α nor β commutes with σ_1.

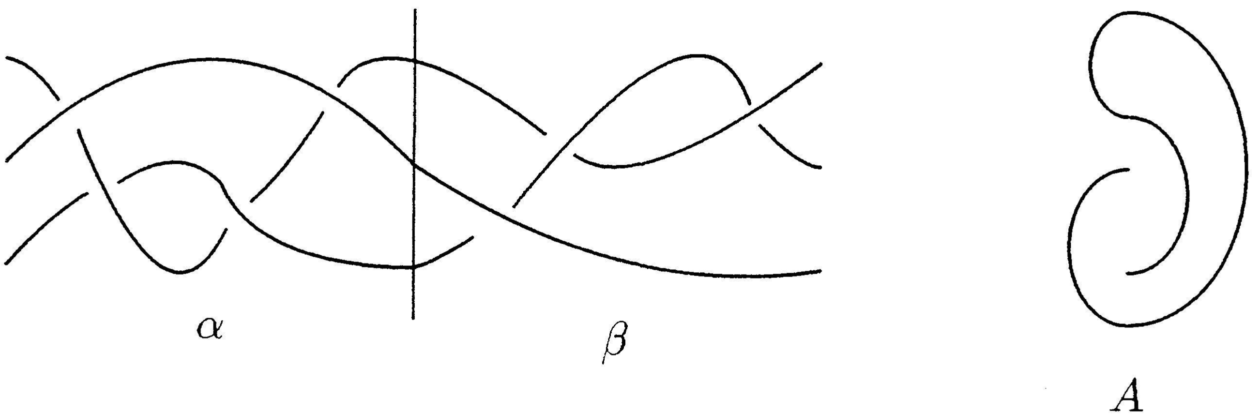

FIGURE 3

The braid $\alpha\beta = \sigma_2^{-1}\sigma_1^2\sigma_2\sigma_1\sigma_2^{-2}$ and the arc $A = [1, 2] * \alpha = \beta * [1, 2]$

We can now formulate the central result of this paper.

2.1. THEOREM. *A braid* $\beta \in B_n$ *commutes with a generator* σ_j *if and only if it has a* (j, j)*-band. Equivalently, the action of* $*\beta$ *on* $\mathbf{A}_n$ *stabilises* $[j, j+1]$. □

This is an immediate corollary of a more general result.

2.2. THEOREM. *For a braid* $\beta \in B_n$ *the following are equivalent:*

(*a*) $\sigma_j\beta = \beta\sigma_k$,

(*b*) $\sigma_j^r\beta = \beta\sigma_k^r$, *for some nonzero integer* r,

(*c*) $\sigma_j^r\beta = \beta\sigma_k^r$, *for every integer* r,

(*d*) β *has a* (j, k)*-band,*

(*e*) $[j, j+1] * \beta = [k, k+1]$.

2.3. COROLLARY. *The centraliser of* σ_j^r *is independent of* $r \neq 0$ *and coincides with the stabiliser of the interval* $[j, j+1]$ *in the action of* B_n *upon* $\mathbf{A}_n$. □

2.4 COROLLARY. *The inner automorphism in* B_n *exchanging generators,* $\sigma_k = \beta^{-1}\sigma_j\beta$, *is achieved exactly by those braids* β *that have a* (j, k)*-band.* □

2.5 COROLLARY [Chow]. *The centre of* B_n, $n \geqslant 3$ *is infinite cyclic, generated by the braid* Δ^2, *where*

$$\Delta = \sigma_{n-1}(\sigma_{n-2}\sigma_{n-1}) \cdots (\sigma_1\sigma_2 \cdots \sigma_{n-1}) .$$

Proof. A braid commutes with all braid generators if and only if its action stabilises all the intervals $[1, 2], \ldots, [n-1, n]$, so it has a great ribbon containing the entire braid, connecting $[1, n] \times 0$ with $[1, n] \times 1$, necessarily in an order-preserving sense. Such a braid is clearly a multiple of the full-twist Δ^2. □

3. PROOF OF THEOREM 2.2

It is useful here to introduce an invariant of proper arcs. Throughout this section A will denote an oriented (k, l)-arc in $\mathbf{C}$ which is proper with respect to $\{1, \ldots, n\}$.

Associated with A is a word in the symbols $I_0, I_1, \ldots, I_n, I_0^{-1}, I_1^{-1}, \ldots, I_n^{-1}$ which can be described as follows. Assume that A is transverse to the real line. Starting from its initial point k, continue along A to l and whenever A crosses the interval $[m, m+1]$ write I_m if it crosses with increasing imaginary part and write I_m^{-1} otherwise. In the above notation, use the interval $(-\infty, 1]$ in case $m = 0$ and $[n, \infty)$ if $m = n$, in place of $[m, m+1]$. An isotopy of A will change the word by a sequence of moves of the following sort:

a) the introduction or deletion of cancelling pairs of the form $I_m I_m^{-1}$ or $I_m^{-1} I_m$,

b) left multiplication by a word in I_{k-1}, I_k and

c) right multiplication by a word in I_{l-1}, I_l.

Let $w(A)$ be the word in the free group on the symbols $I_0, I_1, \ldots, I_n$ obtained by deleting all cancelling pairs, all initial segments in I_{k-1}, I_k and all final segments in I_{l-1}, I_l. Then $w(A)$ is an isotopy invariant, and it is routine to check that A can be isotoped to read off exactly the word $w(A)$. Note that the exponents ± 1 of symbols in $w(A)$ necessarily alternate.