

4. Centralisers of braid subgroups

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iterated $2r$ times. Noting that I_p persists in w^* it is easy to argue that $w(A * \sigma_j^{2r}) = w(A)$ is impossible; the contradiction. \square

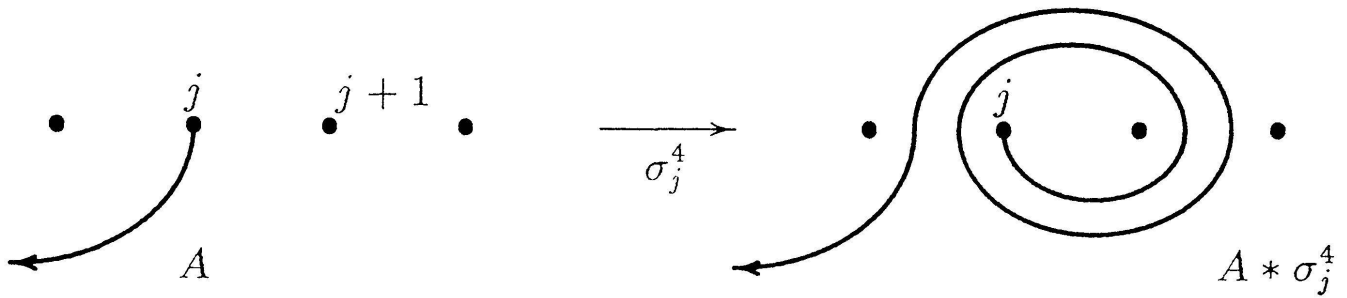


FIGURE 4

The action of $*\sigma_j^{2r}$ on a (j, k) -arc in case $r = 2$

We now turn to the proof of Theorem 2.2. It has already been observed that $(e) \Rightarrow (d) \Rightarrow (a)$, and it is obvious that $(a) \Rightarrow (c) \Rightarrow (b)$. So it remains to establish that $(b) \Rightarrow (e)$. Thus we assume that, for some $r \neq 0$, $\sigma_j^r \beta = \beta \sigma_k^r$. Since the algebraic crossing number of any two strings of a braid is a well-defined braid invariant, this equation is possible only if $\{j, j+1\} * \beta = \{k, k+1\}$. Now, noting that $\beta^{-1} \sigma_j^r \beta = \sigma_k^r$ and that σ_k^r has a (k, k) -band, we conclude that there is a proper ribbon for $\beta^{-1} \sigma_j^r \beta$ from $[k, k+1] \times 0$ to $[k, k+1] \times 1$. Define $A = \beta * [k, k+1] = [k, k+1] * \beta^{-1}$. Then we may assume (possibly after an isotopy) that the planes $\mathbf{C} \times 1/3$ and $\mathbf{C} \times 2/3$ cut the ribbon in the arcs $A \times 1/3$ and $A \times 2/3$. Moreover, the middle third of the ribbon, and Proposition 1.1, imply that $A * \sigma_j^r = A$. By Lemma 3.2, $A = [j, j+1]$ and the theorem is proved. \square

4. CENTRALISERS OF BRAID SUBGROUPS

We have established the following.

4.1 THEOREM. *The centraliser in B_n of the generator σ_j is the subgroup of all braids which have (j, j) -bands. This subgroup is isomorphic to $B_{n-1}^j \times \mathbf{Z}$ where B_{n-1}^j is the subgroup of B_{n-1} consisting of all $(n-1)$ -braids whose permutations stabilise j . \square*

The goal of this section is to describe the centraliser of B_r in B_n , $r \leq n$, which we will call $C(r, n)$. Here B_r is the r -string braid group with its usual inclusion in B_n , namely as the subgroup generated by $\sigma_1 \dots \sigma_{r-1}$.

4.2 THEOREM. *The centraliser $C(r, n)$ of B_r in B_n consists of all n -braids in which the first r strings lie on a ribbon, disjoint from the other strings, and which intersects $\mathbf{C} \times 0$ and $\mathbf{C} \times 1$ in exactly the straight line intervals from $[1, r] \times 0$ and $[1, r] \times 1$ (up to isotopy).*

Proof. A braid β is in $C(r, n)$ if and only if it commutes with each σ_j , $1 \leq j \leq r - 1$. Thus $[j, j + 1] * \beta = [j, j + 1]$, $1 \leq j \leq r - 1$ and so $[1, r] * \beta = [1, r]$, up to isotopy fixing $\{1, \dots, n\}$. \square

It follows that $C(r, n)$ consists of all n -braids constructible as follows. Let $k = n - r + 1$ and consider the subgroup B_k^1 of k -braids whose associated permutation fixes 1. Then replace the first string of a braid in B_k^1 by r parallel strings lying on a ribbon along that string. The ribbon may be twisted by some integral multiple of 2π (or π in the case $r = 2$); such braids are precisely the central elements of B_r .

4.3 THEOREM. *The centraliser $C(r, n)$ is isomorphic to the direct product $B_{n-r+1}^1 \times \mathbf{Z}$.* \square

A PRESENTATION OF $C(r, n)$. In order to establish a set of generators and defining relations for $C(r, n)$ we recall results of Chow [Ch] regarding B_k^1 . This subgroup of B_k is generated by $\sigma_2, \dots, \sigma_{k-1}$, together with elements a_2, \dots, a_k defined by

$$a_i := \sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_{i-2}^{-1} \sigma_{i-1}^2 \sigma_{i-2} \cdots \sigma_2 \sigma_1.$$

These generators satisfy the usual braid relations:

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i, & |i - j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \end{aligned}$$

as well as the following, for $i = 2, \dots, k - 1$:

$$\begin{aligned} \sigma_i a_j \sigma_i^{-1} &= a_j, & j \neq i, i + 1 \\ \sigma_i a_i \sigma_i^{-1} &= a_{i+1} \\ \sigma_i a_{i+1} \sigma_i^{-1} &= a_{i+1}^{-1} a_i a_{i+1}. \end{aligned}$$

In fact these are *defining* relations for B_k^1 . Chow also noted that the subgroup of B_k^1 generated by the a_i is a normal subgroup (as is clear from the above relations), in fact a *free* group on the generators a_i , and that B_k^1 could be regarded as the semidirect product of that free subgroup with the

subgroup generated by $\sigma_2 \dots \sigma_{k-1}$, the latter group clearly isomorphic with the braid group on $k - 1$ strings.

Applying this to our situation, for each $i = 1, \dots, n - r$, let A_{r+i} be the n -braid resulting from replacing the first string of the k -braid a_i , defined above, by r parallel strings which lie on an untwisted band. Specifically,

$$A_{r+i} = (\sigma_r^{-1} \sigma_{r+1}^{-1} \cdots \sigma_{r+i-2}^{-1} \sigma_{r+i-1}) (\sigma_{r-1}^{-1} \sigma_r^{-1} \cdots \sigma_{r+i-3}^{-1} \sigma_{r+i-2}) \cdots (\sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_{i-1}^{-1} \sigma_i) \times (\sigma_i \sigma_{i-1} \cdots \sigma_1) (\sigma_{i+1} \sigma_i \cdots \sigma_2) \cdots (\sigma_{r+i-1} \sigma_{r+i-2} \cdots \sigma_r)$$

Also let C denote the well-known generator of the centre of the r -string braid group, namely $C = \sigma_1$ if $r = 2$ and in case $r > 2$:

$$C = (\sigma_1 \sigma_2 \cdots \sigma_{r-1})^r .$$

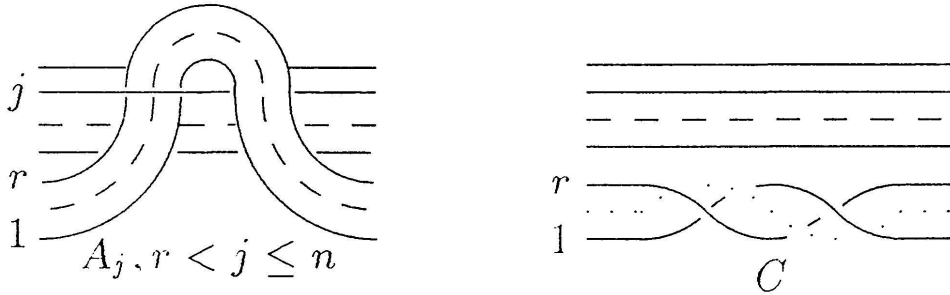


FIGURE 5

Special generators of $C(r, n)$

4.4 THEOREM. *The centraliser $C(r, n)$ of B_r in B_n has the generators:*

$$\sigma_{r+1}, \sigma_{r+2}, \dots, \sigma_{n-1}, A_{r+1}, \dots, A_n, C$$

and defining relations:

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i, & |i - j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i A_j \sigma_i^{-1} &= A_j, & j \neq i, i + 1 \\ \sigma_i A_i \sigma_i^{-1} &= A_{i+1} \\ \sigma_i A_{i+1} \sigma_i^{-1} &= A_{i+1}^{-1} A_i A_{i+1} \\ C \sigma_i &= \sigma_i C \\ C A_i &= A_i C . \end{aligned}$$

(Subscripts ranging over all values for which the symbols are in the list of generators.) \square