3. Functions of excessive growth: the case $f(d) = d^\alpha$

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We choose $T = \varphi_1(x)^{y/4} = (\log x/\varphi(x))^{y/4}$. The upper bound above becomes

$$\ll \varphi(x)^{y/2}(\log x)^{y/2-1}(\log \varphi_1(x))^2 . \tag{56}$$

Indeed the last term is easily seen to be negligible by the lower bound (40) imposed on $\varphi(x)$, and because $R(x) = \exp\{(\log x)^{4/7}\}$ is an admissible choice for R. Thus we may define $E_2(x,y)$ as being equal to a suitable constant multiple of the right-hand side of (56), and apply Theorem 7 to obtain that

$$\Delta(n;f) \ll \xi(n)\tau(n)y^{\Omega(n)/2}\sqrt{E_2(n,y)} \quad \text{pp}l , \tag{57}$$

provided $0 \leqslant y \leqslant y_0 < 4$ and $y = y(n)$ is such that $E_2(n,y)$ is slowly increasing as a function of n. We choose

$$y = 2\Big/\left(1 + \frac{\log\varphi(n)}{\log_2 n}\right) = 1\Big/\left(1 - \frac{\log\varphi_1(n)}{2\log_2 n}\right),$$

which minimises $(\log n)^{-\log y}E_2(n,y)$ up to a power of $\log_2\varphi_1(n)$. This value of y is always in the range $[1,2]$. Inserting into (56) yields

$$E_2(n,y) \asymp (\log\varphi_1(n))^2 ,$$

which implies that this function is slowly decreasing. The required estimate (42) hence follows from (57). This completes the proof of Theorem 10.

3. Functions of excessive growth: the case $f(d) = d^\alpha$

Here, we address the problem of bounding the discrepancy pp$l$ for functions which increase too fast for the techniques of the previous section to be applicable. More precisely, let us recall the quantity

$$H_\nu(x,y) := \sum_{k=1}^{\infty}\left(\frac{y}{4}\right)^{\Omega(k)}\frac{1}{k^{1+\sigma}}\left|\sum_{m=1}^{\infty}\left(\frac{y}{4}\right)^{\Omega(m)}\frac{e(\nu f(km))}{m^{1+\sigma}}\right|^2, \tag{58}$$

with $\sigma := 1/\log x$, which appears implicitly in the upper bound (26) of Theorem 7 for the discrepancy $\Delta(n;f)$. This was primarily defined for $y < 8$, but we restrict if here to values of $y \leqslant 4$. The functions of moderate growth are essentially those for which the inner m-sum can be estimated by partial summation, using the available results on the mean value of $m \mapsto (y/4)^{\Omega(m)}$. When the rate of growth of f prohibits such a treatment, we may consider $H_\nu(x,y)$ as a 'type II sum', according to the poetic terminology of Vinogradov. For all intents and purposes, this means making the trivial estimate $|(y/4)^{\Omega(k)}| \leqslant 1$, expanding the square and, after permuting summations, estimating the inner k-sum by an *ad hoc* exponential sum method.

This programme may be carried out, in principle, for any smooth function f of, say, at most polynomial growth, and indeed one could even aim at a general theorem established along these lines and providing, under suitable sufficient conditions, explicit upper bounds for the discrepancy. Due to the considerable amount of calculations that this would involve, we have preferred to treat only examples which reflect all the difficulties of the general case, but avoid tedious technicalities that would hide the main stream of the argument. In this context, we believe that the functions

$$d \mapsto d^{\alpha} \quad (\alpha \in \mathbf{R}^{+} \setminus \mathbf{Z}^{+}), \qquad d \mapsto \theta d \quad (\theta \in \mathbf{R} \setminus \mathbf{Q})$$

are of special interest. We treat the first of these immediately and the second one in the next section. The following theorem provides an effective version of the corresponding qualitative result obtained by Hall & Tenenbaum in [13].

THEOREM 11. *Let $\alpha > 0$ be a given real number, not an integer. Then the function $g_{\alpha}(d) := d^{\alpha}$ is* erd. *More precisely, we have*

$$\Delta(n; g_{\alpha}) < \tau(n)(\log n)^{-\delta} \quad \text{pp}l \tag{59}$$

for all $\delta < \delta_0 := \frac{1}{2}\log\frac{12}{11}$.

We note that $\delta_0 > \frac{1}{23}$. We have not attempted here to find the best exponent available from latest developments in exponential sums theory and have confined ourselves to using a result of Karatsuba [16] on Vinogradov-type bounds which is expressed in an easily applicable form. We remark that van der Corput-type estimates would in general yield weaker bounds for the discrepancy and would actually only save a power of $\log_2 n$ in (59).

It is also worthwhile to note at this stage that the method of proof of Theorem 11 will readily yield that the function

$$f(d) = \exp\{(\log d)^{\alpha}\}$$

is erd for $0 < \alpha < 1$, and indeed it will provide a bound comparable with (59) for the discrepancy. In particular, this shows that the limitation $\alpha < 3/5$ which arises from mere application of Theorems 4 or 10 is purely technical. The range $1 < \alpha < 3/2$ may also be handled by the same technique, but with a weaker effective result — see Theorem 2 of Karatsuba [16].

By Theorem 3, we immediately derive from the above result the following corollary.

COROLLARY 5. *Let* α, δ *be as in the statement of Theorem 11. Then the sequence*

$$\{n \geqslant 2 : \langle n^{\alpha} \rangle \leqslant (\log n)^{-\delta}\} \tag{60}$$

is a Behrend sequence.

Of course we can rewrite the condition in (60) introducing $j := [n^{\alpha}]$. This yields the following reformulation in terms of block sequences.

COROLLARY 6. *Let* $\beta > 0$, $1/\beta \notin \mathbf{Z}$, $\delta < \delta_0 = \frac{1}{2}\log\frac{12}{11}$. *Then the sequence*

$$\mathscr{B}(\delta) := \bigcup_{j=2}^{\infty} \left] j^{\beta}, j^{\beta}\left(1 + \frac{1}{j(\log j)^{\delta}}\right)\right] \cap \mathbf{Z}^{+} \tag{61}$$

is a Behrend sequence.

As far as block sequences are concerned, this is only significant when $\beta > 1$: otherwise the 'blocks' have lengths smaller than 1 and looking at $\mathscr{B}(\delta)$ as a block sequence is meaningless. As we remarked in the previous section, the above result is unreachable, in the present state of knowledge, by the technique applied in [24]. The natural conjecture in accord with the results of [15] and [24] would be that $\mathscr{B}(\delta)$ is Behrend for all $\delta < \log 2$, this exponent then being optimal. This is also out of reach of the present technique, which implies a systematic loss due, among other causes, to the trivial estimate for $(y/4)^{\Omega(k)}$ in (58).

We now embark on the proof of Theorem 11. We give ourselves two parameters x_1, x_2 satisfying

$$e^{\sqrt{\log x}} \leqslant x_1 \leqslant x, \qquad x_2 := x^{\log_2 x},$$

and introduce the following further notation

$$J = J(x) := \frac{\log(x_2/x_1)}{\log 2}, \qquad K_j := 2^j x_1 \quad (0 \leqslant j \leqslant J),$$

$$B_j(m, n; \nu, f) := \sum_{K_j < k \leqslant K_{j+1}} e(\nu f(kn) - \nu f(km)) \quad (0 \leqslant j \leqslant J).$$

LEMMA 1. *Let* $\alpha > 0$. *Then we have uniformly for* $x \geqslant 3, \nu \geqslant 1, 0 \leqslant y \leqslant 4$, *and a real valued arithmetical function* f

$$H_\nu(x, y) \ll (\log x)^{y/2} (\log x_1)^{y/4} + \sum_{\substack{x_1 < m < n \leqslant x_2 \\ n^\alpha - m^\alpha \geqslant 1}} \frac{\left(\frac{1}{4} y\right)^{\Omega(mn)}}{mn} \sum_{0 \leqslant j \leqslant J} \frac{\Re e B_j(m, n; \nu, f)}{K_j} . \tag{62}$$

Proof. To lighten the presentation, we temporarily set $z := y/4$. We first split the k-sum in (58) according to whether $k \leqslant x_1$, $x_1 < k < x_2$ or $k > x_2$, so as to write correspondingly

$$H_\nu(x, y) = H_\nu^{(1)}(x, y) + H_\nu^{(2)}(x, y) + H_\nu^{(3)}(x, y) .$$

Using the bounds

$$\sum_{k \leqslant x_1} z^{\Omega(k)} k^{-1} \ll (\log x_1)^z , \tag{63}$$

and

$$\sum_{k > x_2} z^{\Omega(k)} k^{-1-\sigma} \leqslant x_2^{-\sigma/2} \sum_{k \geqslant 1} z^{\Omega(k)} k^{-1-\sigma/2} \ll (\log x)^{z-1/2} , \tag{64}$$

we readily obtain

$$\begin{aligned} H_\nu^{(1)}(x, y) + H_\nu^{(3)}(x, y) &\ll (\log x_1)^z (\log x)^{2z} + (\log x)^{3z-1/2} \\ &\ll (\log x)^{y/2} (\log x_1)^{y/4} . \end{aligned} \tag{65}$$

Next, we split the inner m-sum in $H_\nu^{(2)}(x, y)$ at x_1 and x_2 and use the inequality $(a + b + c)^2 \leqslant 3(a^2 + b^2 + c^2)$ to obtain

$$H_\nu^{(2)}(x, y) \leqslant 3H_\nu^{(21)}(x, y) + 3H_\nu^{(22)}(x, y) + 3H_\nu^{(23)}(x, y)$$

with

$$\begin{aligned} H_\nu^{(21)}(x, y) &\leqslant \sum_{x_1 < k \leqslant x_2} z^{\Omega(k)} k^{-1} \left(\sum_{m \leqslant x_1} z^{\Omega(m)} m^{-1} \right)^2 \\ &\ll (\log x)^z (\log x_1)^{2z} \ll (\log x)^{y/2} (\log x_1)^{y/4} , \end{aligned} \tag{66}$$

by (63), and

$$\begin{aligned} H_\nu^{(23)}(x, y) &\leqslant \sum_{x_1 < k \leqslant x_2} z^{\Omega(k)} k^{-1} \left(\sum_{m > x_2} z^{\Omega(m)} m^{-1-\sigma} \right)^2 \\ &\ll (\log x)^{3z-1} = (\log x)^{3y/4-1} \ll (\log x)^{y/2} , \end{aligned} \tag{67}$$

by (64).

It remains to estimate $H_\nu^{(22)}(x, y)$. We have

$$(68)\quad H_\nu^{(22)}(x,y) \leqslant \sum_{0 \leqslant j \leqslant J} K_j^{-1} \sum_{K_j < k \leqslant K_{j+1}} \Big| \sum_{x_1 < m \leqslant x_2} \frac{z^{\Omega(m)}}{m^{1+\sigma}} e(\nu f(km)) \Big|^2 ,$$

where we have made the trivial estimate $z^{\Omega(k)} \leqslant 1$. Expanding the square, we find that it does not exceed

$$(69)\quad 2\Re e \sum_{\substack{m < n \leqslant x_2 \\ n^\alpha - m^\alpha \geqslant 1}} \frac{z^{\Omega(mn)}}{(mn)^{1+\sigma}} e(\nu f(kn) - \nu f(km)) + 2 \sum_{\substack{x_1 < m \leqslant n \leqslant x_2 \\ n^\alpha - m^\alpha < 1}} \frac{1}{mn} .$$

We claim that the second sum on the right is $\ll x_1^{-\min(1,\alpha)}$. This plainly holds if $\alpha \geqslant 1$ since the summation conditions then imply that $m = n$. When $0 < \alpha < 1$, we note that $n^\alpha - m^\alpha \geqslant \alpha(n-m)n^{\alpha-1}$, so for fixed n the m-sum is $\ll n^{1-\alpha}/n = n^{-\alpha}$ and the conclusion is still valid. Inserting this estimate into (69) and (68) and using the fact that $J \ll x_1^{\min(1,\alpha)}$, we obtain

$$H_\nu^{(22)}(x,y) \ll \Re e \sum_{0 \leqslant j \leqslant J} \frac{1}{K_j} \sum_{K_j < k \leqslant K_{j+1}} \sum_{\substack{x_1 < m < n \leqslant x_2 \\ n^\alpha - m^\alpha \geqslant 1}} \frac{z^{\Omega(mn)}}{(mn)^{1+\sigma}} e(\nu f(kn) - \nu f(km)) + 1 .$$

We permute summations on k and m, n and see that the new, inner k-sum equals $B_j(m, n; \nu, f)$. Together with (65), (66) and (67), this completes the proof of our lemma.

We now apply Karatsuba's estimate to bound the exponential sum

$$(70)\quad B(K; \upsilon) := \sum_{K < k \leqslant 2K} e(\upsilon k^\alpha)$$

for relevant values of υ, K.

LEMMA 2. *Let* $\alpha \in \mathbf{R}^+ \setminus \mathbf{Z}^+$. *There exists a constant* $c = c(\alpha) > 0$ *such that the estimate*

$$(71)\quad B(K; \upsilon) \ll K^{1-c} + K \exp\left\{ -c \frac{(\log K)^3}{(1 + \log \upsilon)^2} \right\}$$

holds uniformly for $K \geqslant 1, \upsilon \geqslant 1$.

Proof. If $\upsilon \leqslant K^{1-\alpha}$, and so $0 < \alpha < 1$, we apply a classical estimate of van der Corput (see e.g. Titchmarsh [26], lemma 4.7) to get

$$B(K; \upsilon) = \int_K^{2K} e(\upsilon t^\alpha) \mathrm{d}t + O(1) \ll K^{1-\alpha} .$$

(The same estimate also follows from theorem 2.1 of Graham & Kolesnik [7].) Thus (71) holds in this case.

If $\upsilon > K^{1-\alpha}$, set $n := [3\alpha + 3(\log \upsilon)/\log K]$, so that $n \geqslant 3$ and

$$K^{n/3} \leqslant \upsilon K^{\alpha} < K^{(n+1)/3} .$$

Put $g(t) := \upsilon t^{\alpha}$. We have for all non-negative integers s

$$\frac{g^{(s)}(t)}{s!} = \upsilon \binom{\alpha}{s} t^{\alpha - s} .$$

Writing $\binom{\alpha}{s} = (-1)^s \prod_{1 \leqslant j \leqslant s}\{1 - (\alpha+1)/j\}$, we see that we have for suitable positive constants $c_1 = c_1(\alpha)$, $c_2 = c_2(\alpha)$,

$$c_1 s^{-\alpha-1} \leqslant \left|\binom{\alpha}{s}\right| \leqslant c_2 s^{-\alpha-1} \qquad (s \geqslant 0) . \tag{72}$$

Hence for large K and $K \leqslant t \leqslant 2K$ we have

$$\left|\frac{g^{(n+1)}(t)}{(n+1)!}\right| \leqslant \frac{2^{\alpha} c_2}{(n+1)^{\alpha+1}} \upsilon K^{\alpha - n - 1} \leqslant 2^{\alpha} c_2 K^{-2(n+1)/3} \leqslant K^{-(n+1)/2} .$$

Similarly, a straightforward computation enables us to deduce from (72) that for all s in the range $3n/4 \leqslant s \leqslant n$ (so $s \geqslant 3$) and large K we have

$$K^{-3s/4} \leqslant c_1 s^{-\alpha-1} 2^{-s} K^{-2/3} \leqslant \left|\frac{g^{(s)}(t)}{s!}\right| \leqslant 2^{\alpha} K^{-5s/9 + 1/3} \leqslant K^{-s/3} .$$

By Theorem 1 of Karatsuba [16], it follows that, for suitable positive absolute constants c_3 and c_4 and $K > K_0(\alpha)$ we have

$$|B(K;\upsilon)| \leqslant c_3 K^{1 - c_4/n^2} .$$

This implies the required bound.

We are now in a position to complete the proof of Theorem 11. We want to apply Theorem 7 and use Lemmas 1 and 2 to obtain an upper bound for the quantity $E_2(n,y)$. We select, in Lemma 1, $f = g_{\alpha}$, as defined in the statement of the theorem, and

$$x_1 = \exp\{c(\log x)^{2/3} \log_2 x\} ,$$

with $c = c(\alpha) > 0$ at our disposal. Then, with the notation of Lemma 2, we have $B_j(m,n;\nu,g_a) = B(K_j;\upsilon)$ where $\upsilon := \nu(n^{\alpha} - m^{\alpha})$, so $\upsilon \leqslant x_2^{1+\alpha}$ provided $\nu \leqslant x$. By Lemma 2 there is a positive constant $c_5 = c_5(\alpha)$ such that, for all $m, n \leqslant x$ with $n^{\alpha} - m^{\alpha} \geqslant 1$,

$$B_j(m, n; \nu, g_a)/K_j \ll \exp\left\{-c_5 \frac{(\log K_j)^3}{(\log x_2)^2}\right\} \ll 1/(\log x)^4$$

provided $c(\alpha)$ is large enough. By (62), we infer that we have uniformly for $1 \leqslant \nu \leqslant x$, $0 \leqslant y \leqslant 4$,

$$H_\nu(x, y) \ll (\log x)^{2y/3} (\log_2 x)^{y/4} . \tag{73}$$

Inserting the above estimate into (24) with, say, $T := \log x$, we see that we may choose

$$E_2(x, y) \asymp (\log x)^{11y/12 - 1} (\log_2 x)^{2 + y/4} , \tag{74}$$

which is hence slowly increasing. Therefore we get by Theorem 7 that

$$\Delta(n, g_\alpha) < \tau(n) (\log n)^{11y/24 - 1/2 - (1/2)\log y + o(1)} \qquad \mathrm{pp}l . \tag{75}$$

The required estimate (59) now follows on taking optimally $y = \frac{12}{11}$.

4. Functions of excessive growth: the case $f(d) = \theta d$

We now investigate, in a quantitative form, the uniform distribution on divisors of the function

$$h_\theta(d) := \theta d$$

when θ is a given irrational real number. This study is similar in principle to that of the previous section, but more complicated inasmuch as the effective bounds for $\Delta(n; h_\theta)$ will depend on the arithmetic nature of θ. On the other hand we shall not need, as might be expected, any involved tool for the estimation of the relevant exponential sums.

More explicitly, let us define $Q(x) := x/(\log x)^{10}$, and

$$q(x; \theta) := \inf\{q : 1 \leqslant q \leqslant Q(x),\ \| q\theta \| \leqslant 1/Q(x)\} \tag{76}$$

where $\| u \|$ denotes the distance of u to the set of integers. Our results depend on a free parameter $y, 0 < y \leqslant 4$, and may be expressed conveniently in terms of any increasing lower bound for $q(x; \theta)$, say $q^*(x; y, \theta)$, with the property that $q^*(x; y, \theta)/(\log x)^{y/4}$ is decreasing. A possible choice is

$$q^*(x; y, \theta) := 4(\log x)^{y/4} \Big/ \int_1^x \frac{y(\log t)^{y/4 - 1}}{t \inf_{u \geqslant t} q(u; \theta)} \, \mathrm{d}t . \tag{77}$$

Unless θ has abnormally good rational approximations, we have

$$q^*(x; y, \theta) \asymp (\log x)^{y/4} . \tag{78}$$