§5. Additive functions

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§5. Additive functions

As we noted in the first section, the study of uniform distribution on divisors of additive functions is made much easier by the fact that the Weyl sums are multiplicative. Indeed this is the only case when we are able to achieve estimates for the discrepancy which go beyond the statistical bound $\sqrt{\tau(n)}$.

We shall prove the following theorem, which provides a simple, but nevertheless effective criterion. We write for integer ν

$$(102)\qquad L_\nu(x;f) := \sum_{p \leqslant x} \frac{\| \nu f(p) \|^2}{p}, \quad C_\nu(x;f) = \sum_{p \leqslant x} \frac{1 - |\cos \nu\pi f(p)|}{p},$$

and note that we have for all x

$$(103)\qquad 4L_\nu(x;f) \leqslant C_\nu(x;f) \leqslant \tfrac{1}{2}\pi^2 L_\nu(x;f)\,.$$

Theorem 13. *Let f be an additive function. Then f is* erd *if, and only if,*

$$(104)\qquad \sum_p \frac{\| \nu f(p) \|^2}{p} = \infty \qquad (\nu \neq 0)\,.$$

Moreover, if this is the case then we have

$$(105)\qquad \sum_{n \leqslant x} \frac{\Delta(n;f)}{n} \left(\frac{y}{2}\right)^{\Omega(n)} \ll (\log x)^y \left\{ \frac{1}{T} + \sum_{1 \leqslant \nu \leqslant T} \frac{e^{-yC_\nu(x;f)}}{\nu} \right\}$$

uniformly for $x \geqslant 2$, $T \geqslant 1$, $0 \leqslant y \leqslant y_0 < 4$.

The qualitative result corresponding to this statement (i.e. the criterion (104)) was first established by Kátai [18].

By (104) and the lower bound of (103), the expression between curly brackets tends to zero for suitable $T = T(x,y)$. Hence (105) provides, as described in Theorem 6, a family of effective pp$l$ estimates for $\Delta(n;f)$ which may of course be further optimised with respect to y. This process must lead to a non trivial result since $y = 1$ is admissible.

We give below two applications of Theorem 13, respectively devoted to the functions $f(d) = \theta\Omega(d)$ for irrational θ, and $f(d) = \log d$. Recovering a result first obtained independently by Hall and Kátai, the latter case furnishes the best known pp$l$ upper bound for the discrepancy for any function we are aware of, although this falls short of the current conjecture stated in section 2. The former case provides an example of a function which is erd but has the same rate of growth than, and indeed is asymptotically equal to, a function which is not, namely $\theta \log_2 n$ — see Corollary 3. Actually, if we define $\Omega(n; E)$ as the number of those prime factors of n which belong to some set of primes E, then Theorem 13 implies that $\theta\Omega(n; E)$ is erd for all $\theta \in \mathbf{R}\backslash\mathbf{Q}$ if, and only if, $\sum_{p \in E} 1/p = \infty$. We thus exhibit erd additive functions with an arbitrarily slow growth.

The effective bounds for $\Delta(n; \theta\Omega)$ naturally depend on the rational approximations to θ. We set $Q_1(x) := \sqrt{\log_2 x/\log_3 x}$, and define

$$q_1(x;\theta) := \inf_{t \geqslant x} \max\{q \leqslant Q_1(t) : \| \theta q \| \leqslant 1/Q_1(t)\} . \tag{106}$$

Then $q_1(x;\theta)$ increases to ∞ for all irrational θ and furthermore $q_1(x;\theta) = (\log_2 x)^{1/2+o(1)}$ for almost all θ.

COROLLARY 10. *Let* $\theta \in \mathbf{R}\backslash\mathbf{Q}$. *Then the function* $\theta\Omega(n)$ *is* erd *and we have*

$$\Delta(n; \theta\Omega) < \tau(n) q_1(n;\theta)^{-1+o(1)} \quad \text{pp}l\,. \tag{107}$$

COROLLARY 11 (Hall [8]; Kátai [17]). *Let* $\alpha > (\log(4/\pi))/\log 2 \approx 0 \cdot 34850$. *We have*

$$\Delta(n; \log) < \tau(n)^{\alpha} \quad \text{pp}l\,. \tag{108}$$

We now embark on the proof of Theorem 13. That the condition is necessary is a straightforward consequence of the definition of uniform distribution on divisors in the form (1). Indeed, suppose that (104) fails to hold for $\nu \neq 0$. Writing $F(z; n) := \sum_{d \mid n, \langle f(d) \rangle \leqslant z} 1$, we have

$$g_\nu(n) := \sum_{d \mid n} e(\nu f(n)) = \int_0^1 e(\nu z) \mathrm{d}F(z; n) = \int_0^1 e(\nu z) \mathrm{d}(F(z; n) - z\tau(n))$$

$$= 2\pi i \nu \int_0^1 e(\nu z)(F(z; n) - z\tau(n)) \mathrm{d}z\,,$$

hence

$$2\pi \mid \nu \mid \Delta(n; f) \geqslant \mid g_\nu(n) \mid . \tag{109}$$

Now for all $\varepsilon \in]0, 1[$, we have

$$\begin{aligned} \sum_{n \leqslant x} \frac{\mu(n)^2 \mid g_\nu(n) \mid}{n\tau(n)} &\geqslant \sum_{P^+(n) \leqslant x^\varepsilon} \frac{\mu(n)^2 \mid g_\nu(n) \mid}{n\tau(n)} - \sum_{\substack{n > x \\ P^+(n) \leqslant x^\varepsilon}} \frac{1}{n} \\ &= \prod_{p \leqslant x^\varepsilon} \left(1 + \frac{\mid \cos \pi\nu f(p) \mid}{p}\right) - O(e^{-1/2\varepsilon} \log x)\,, \end{aligned} \tag{110}$$

where the O-estimate follows from (94) by partial summation. The product is

$$\gg \exp\left\{\sum_{p\leqslant x^{\varepsilon}} \frac{|\cos\pi\nu f(p)|}{p}\right\} \geqslant \exp\left\{\sum_{p\leqslant x^{\varepsilon}} \frac{1-\frac{1}{2}\pi^2\|\nu f(p)\|^2}{p}\right\} \gg \varepsilon\log x\,,$$

by our assumption that the series (104) converges. Inserting this into (110) and choosing ε small enough but fixed, we obtain that the left-hand side is $\gg \log x$, whence by (109)

$$\sum_{n\leqslant x} \frac{\Delta(n;f)}{n\tau(n)} \gg \log x\,.$$

This cannot hold if $\Delta(n;f) = o(\tau(n))$ pp (or even pp$l$), and we obtain the required necessity assertion.

The sufficiency part of the theorem readily follows from the upper bound (21) which we recall for convenience: we have uniformly for $x \geqslant 2$, $T \geqslant 1$, $0 \leqslant y \leqslant y_0 < 4$,

$$\sum_{n\leqslant x} \frac{\Delta(n;f)}{n}\left(\frac{y}{2}\right)^{\Omega(n)} \ll \frac{(\log x)^y}{T} + \sum_{1\leqslant\nu\leqslant T}\frac{1}{\nu}\sum_{n\leqslant x}\frac{|g_\nu(n)|}{n}\left(\frac{y}{2}\right)^{\Omega(n)}. \tag{111}$$

The inner n-sum on the right does not exceed

$$\sum_{P^+(n)\leqslant x} \frac{|g_\nu(n)|}{n}\left(\frac{y}{2}\right)^{\Omega(n)} \ll \exp\left\{\sum_{p\leqslant x}\frac{y|\cos\pi\nu f(p)|}{p}\right\} \ll (\log x)^y e^{-yC_\nu(x;f)}\,.$$

Inserting this into (111) yields (105). Taking $y = 1$ implies $\Delta(n;f) = o(2^{\Omega(n)})$ pp$l$, from which we deduce in turn that $\Delta(n;f) = o(\tau(n))$ pp$l$ by a familiar argument. Corollary 1 hence implies that f is erd, and this finishes the proof of Theorem 13.

Proof of Corollary 10. We apply the upper bound (105) of Theorem 13 with $y = 1$. We have

$$C_\nu(x;\theta\Omega) = (1 - |\cos\pi\nu\theta|)\log_2 x + O(1)$$

and need a lower bound for this. For given x, there exist integers a, q with $(a,q) = 1$, $q \leqslant Q_1(x) = \sqrt{\log_2 x/\log_3 x}$, and $|\theta - a/q| \leqslant 1/qQ_1(x)$. Furthermore, we have

$$q \geqslant q_1(x;\theta)\,, \tag{112}$$

where the right-hand side is defined by (106). We select $T := q/\log q$, and note that for $\nu \leqslant T$ we have $|\theta\nu - a_\nu/q_\nu| \leqslant T/qQ_1(x) \leqslant 1/q\log q$, with $a_\nu = a\nu/(q,\nu)$, $q_\nu = q/(q,\nu)$. This implies $\|\theta\nu\| \geqslant (1/q) - 1/q\log q$, hence, for large x,

$$1 - |\cos\pi\nu\theta| \geqslant \left(\tfrac{1}{2}\pi^2 + o(1)\right)/q^2 > (\log_3 x)/\log_2 x\,.$$

Inserting this into (105), we arrive at

$$(\log x)^{-1} \sum_{n \leqslant x} \frac{\Delta(n; \theta\Omega)}{n 2^{\Omega(n)}} \ll \frac{1}{T} + \frac{\log q}{\log_2 x} \ll q_1(x; \theta)^{-1+o(1)} .$$

Since $q_1(x;\theta)$ is a non-decreasing function of x, this implies (107) and the proof is thereby completed.

Proof of Corollary 11. Put $\tau(n, \theta) := \sum_{d \mid n} d^{i\theta}$. When $f = \log$, we have that $g_\nu(n) = \tau(n, 2\pi\nu)$. By lemma 30.2 of [14] we infer that, uniformly for $1 \leqslant |\theta| \leqslant \exp\sqrt{\log x}$,

$$\sum_{p \leqslant x} \frac{|\tau(p, \theta)|}{p} = \sum_{p \leqslant x} \frac{\left|\cos\left(\frac{1}{2}\theta \log p\right)\right|}{p} = \frac{2}{\pi} \log x + O(1) .$$

This is proved by partial summation from a strong form of the prime number theorem. Thus we obtain that we have uniformly for $1 \leqslant \nu \leqslant \log x$

$$C_\nu(x; \log) = (1 - 2/\pi) \log_2 x + O(1) .$$

Inserting this into (105) with $T = \log x$ and choosing optimally $y = \pi/2$, we obtain

$$\Delta(n; f) < \xi(n) (\log_2 n) \left(\frac{4}{\pi}\right)^{\Omega(n)} \quad \text{pp}l$$

for all $\xi(n) \to \infty$. This implies the required result by a now standard device.

6. Metric results

In this last section, we investigate the problem of uniform distribution on divisors from a further statistical point of view, regarding as random not only the integer n but also the function f. Thus, we define a measure μ on the set $\mathbf{A}$ of all real valued arithmetical function as the inverse image of the Haar measure on the compact group $(\mathbf{R}/\mathbf{Z})^{\mathbf{N}}$ by the canonical mapping $f \mapsto \langle f \rangle$. In other words, μ is characterised by the property that for all finite families $\{E_j : 1 \leqslant j \leqslant k\}$ of measurable subsets of the torus $\mathbf{R}/\mathbf{Z}$ and for all integers $n_1, n_2, \ldots, n_k$, we have

$$\mu\{f \in \mathbf{A} : \langle f(n_j) \rangle \in E_j \ (1 \leqslant j \leqslant k)\} = \prod_{j=1}^{k} \lambda(E_j) ,$$