

# Introduction

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ON THE GAUSS-BONNET FORMULA  
FOR LOCALLY SYMMETRIC SPACES OF NONCOMPACT TYPE

by Enrico LEUZINGER

ABSTRACT. Let  $X$  be a Riemannian symmetric space of noncompact type and rank  $\geq 2$  and let  $\Gamma$  be a non-uniform, irreducible lattice in the group of isometries of  $X$ . A Gauss-Bonnet formula for the locally symmetric quotient  $V = \Gamma \backslash X$  was first proved by G. Harder. We present a new simple proof which is based on an exhaustion of  $V$  by Riemannian polyhedra with uniformly bounded second fundamental forms.

INTRODUCTION

The generalized Gauss-Bonnet theorem of C.B. Allendoerfer, A. Weil and S.S. Chern asserts that the Euler characteristic of a *closed* Riemannian manifold  $(M, g)$  is given by

$$\chi(M) = \int_M \omega_g$$

where the Gauss-Bonnet-Chern form  $\omega_g = \Psi_g dv_g$  is (locally) computable from the metric  $g$  (see [AW], [C]).

In several articles J. Cheeger and M. Gromov investigated the Gauss-Bonnet theorem for *open* complete Riemannian manifolds with bounded sectional curvature and finite volume. They in particular showed that such manifolds  $M^n$  admit an exhaustion by compact manifolds with smooth boundary,  $M_i^n$ , such that  $\text{Vol}(\partial M_i^n) \rightarrow 0$  ( $i \rightarrow \infty$ ) and for which the second fundamental forms  $\text{II}(\partial M_i^n)$  are uniformly bounded (see [CG1], [CG2], [CG3] and also [G] 4.5.C). By a formula of Chern one has  $\chi(M_i^n) = \int_{M_i^n} \omega_g + \int_{\partial M_i^n} \eta_i$  where  $\eta_i$  is a certain form on the boundary  $\partial M_i^n$  (see [C]). The above two properties imply that  $\lim_{i \rightarrow \infty} \int_{\partial M_i^n} \eta_i = 0$  and hence  $\chi(M_i^n) = \int_{M^n} \omega_g$  for sufficiently large  $i$ . As a consequence the Gauss-Bonnet theorem holds whenever  $\chi(M_i^n) = \chi(M^n)$  for all sufficiently large  $i$ .

We now consider a Riemannian symmetric space  $X$  of noncompact type and rank  $\geq 2$  and a non-uniform, torsion-free lattice  $\Gamma$  in the group of isometries of  $X$ . The quotient  $V = \Gamma \backslash X$  is a locally symmetric space with bounded non-positive sectional curvature and finite volume. Locally symmetric spaces thus provide important examples for the above class considered by Cheeger and Gromov. If  $\Gamma$  is irreducible a remarkable theorem of G. A. Margulis asserts that  $\Gamma$  is *arithmetic* (see [Z], Ch. 6). For quotients of such lattices the Gauss-Bonnet formula was first proved by G. Harder (see [H]). Following M. S. Raghunathan [R1] he explicitly constructed a smooth exhaustion function  $h$  on  $V$  which has no critical points outside a compact set. A certain defect of the function  $h$ , however, is the quite complicated geometry of its sublevel sets (their second fundamental forms, for instance, are not uniformly bounded). As a consequence the proof given in [H] involves rather long and technical estimates.

The purpose of the present note is two-fold. On the one hand to give a new, more geometric proof of the Gauss-Bonnet theorem for locally symmetric spaces, which avoids the technically complicated estimates of [H]. And, on the other hand, to provide an explicit (and independent) illustration of general results in [CG3].

Our approach is based on an exhaustion  $V = \bigcup_{s \geq 0} V(s)$  of locally symmetric spaces *not* by *smooth* submanifolds but by *polyhedra*, i.e. compact submanifolds with corners (see [L2]). The corners which appear here are naturally related to the geometry of  $V$  at infinity (and therefore should not be smoothed). Moreover, for each  $s \geq 0$  the polyhedron  $V(s)$  is a strong deformation retract of  $V$  (see [L3]). The essential new feature of this exhaustion is that the boundaries of  $\partial V(s)$  consist of subpolyhedra of  $V(s)$  which are projections of pieces of horospheres in  $X$ . As a consequence their second fundamental forms are uniformly bounded. This property together with the generalized Gauss-Bonnet formula for Riemannian polyhedra of Allendoerfer-Weil and Chern leads to a considerably simplified new proof of the Gauss-Bonnet theorem for locally symmetric spaces (see Theorem 4.1).

NOTATION. Explicit constants are irrelevant for our purpose. If  $f$  and  $g$  are positive real valued functions on a set  $S$  we thus simply write  $f \prec g$  if there is a constant  $c > 0$  such that  $f(s) \leq cg(s)$  for all  $s \in S$ .