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by Andreas M. HINZ

ABSTRACT. The sequence of moves in the optimal solution of the Tower of Hanoi with an arbitrary number of discs has recently been shown to lead to an example of an infinite square-free string over a six-letter alphabet by recourse to the theory of iterated morphisms. We present a direct approach to this result, using only properties of the Tower of Hanoi itself, which also reveals an implicit infinite square-free string with just five letters.

0. SQUARE-FREE STRINGS AND THE OLIVE SEQUENCE

Suppose you dispose of a reservoir of *letters* (or *symbols*), i.e. an at most countably infinite set $\mathcal{A}$, called an *alphabet*, and you are asked to construct a *word* (or *string*) of infinite length, i.e. a sequence $a \in \mathcal{A}^{\mathbf{N}}$, which does not contain any non-trivial immediate repetition, or *square*, i.e. there are no $m \in \mathbf{N}_0$ and $l \in \mathbf{N}$ such that

$$\forall k \in \{m+1, \dots, m+l\} : a_{k+l} = a_k .$$

(For a concise survey on *square-free* words and their use in mathematics see [4].)

Assume $\mathcal{A} = \mathbf{N}$ and try an apparently economic approach, namely choose a_k as the smallest positive integer such that $(a_1, \dots, a_k)$ does not contain any square. Then you come up with the following sequence:

$$(1,2,1,3,1,2,1,4,1,2,1,3,1,2,1,5,1,2,1,3,1,2,1,4,1,2,1,3,1,2,1,\dots)$$

A connoisseur of the mathematical theory of the Tower of Hanoi (TH) puzzle (see [13] for a comprehensive survey) will immediately recognize the pattern of the disc numbers in the moves of the optimal solution to transfer a perfect tower of 5 discs from one peg to another. The, in fact unique, optimal solution of finite length $2^n - 1$ for the TH with an arbitrary number $n \in \mathbf{N}_0$ of discs has been excessively discussed in particular in the computer science literature — so excessively that an editor asked for "no more articles on this for a while".

On the other hand, apart from its playful appeal (cf. [11]), an object attracts the attention of mathematicians as soon as infinity is involved. A model of the TH with infinitly many discs was fundamental, for instance, in the discovery of the value of the average distance on the Sierpiński gasket (see [15] and [18]). So how do we define an infinite "optimal" sequence of moves for a TH with an inexhaustible provision of ever increasing discs, numbered $1, 2, \ldots$? (A rather bold assumption given that in the original description of the puzzle (see [7]) the discs were made of pure gold !) It can be done by recourse to one of the oldest observations about the finite optimal solution for n discs (see [8]), namely that the smallest disc 1 moves in every odd numbered move, always cyclically from peg 0 through peg 1 to peg 2, say, and that the even moves are then completely determined by the *divine rule* never to place a larger disc on a smaller one. This will take a tower of n discs from peg 0 to peg 1, if n is odd, and from peg 0 to peg 2, if n is even (cf. [13, Proposition 3]). We may therefore adopt this as a *definition* of the TH sequence or *Olive sequence* o (after its discoverer: Raoul Maurice Olive (1865– ?) was a nephew of the inventor of the TH, Édouard Lucas (cf. [14, Section 2], [12]), and at that time a student at the Lycée Charlemagne in Paris), as I prefer to call it to make a distinction from other sequences derived from the TH which we will encounter in the course of this note. The only ambiguity of this convention lies in fixing the direction the smallest disc moves in, and which will become apparent in some asymmetry of the results we will obtain. The Olive sequence is characterized, in fact overdetermined, by the triples $(d_\mu; i_\mu, j_\mu)$, $\mu \in \mathbf{N}$ being the move number, with $d_\mu \in \mathbf{N}$ the disc moving from $i_\mu \in \{0, 1, 2\}$ to $j_\mu \in \{0, 1, 2\}$.

Now the following is easy to prove, for instance by observing that the even moves form an Olive sequence too (cf. also [13, Proposition 1 (o)]): if $\mu = 2^r(2k+1)$, with $r, k \in \mathbf{N}_0$, then $d_\mu = r + 1$, and $(d_\mu)_{\mu \in \mathbf{N}}$ is just the sequence with which we started our discussion. We will now show that it is square-free:

THEOREM 0. *The sequence* $g := (d_\mu)_{\mu \in \mathbf{N}}$ *of the discs moving in* o *represents a square-free string over the infinite alphabet* $\mathbf{N}$. □

Proof. Assume a square (of length $2l$) starts at position $m+1$, i.e.

$$\forall \nu \in \{m+1, \ldots, m+l\} : d_{\nu+l} = d_\nu ;$$

then l is necessarily even, since $d_\mu = 1$ if and only if μ is odd. So we may apply the general rule that $d_{2\mu} = d_\mu + 1$ to those of the νs which are in even positions and arrive at

$$\forall \nu \in \left\{ \frac{m'}{2} + 1, \ldots, \frac{m'}{2} + \frac{l}{2} \right\} : d_{\nu+l/2} = d_\nu$$

with m' either m or $m-1$, depending on the parity of m.

This yields a square of half the length of the original one, such that we will finally end up with a square for which $l = 1$, a case we have already ruled out previously. □

REMARKS.

1. This proof is a nice example of the method of *infinite descent*, frequently employed by Fermat (cf. [5, p. 387]) and representing an early instance of the principle of *mathematical induction*.

2. The letter g denoting the sequence in Theorem 0 stands for either Louis Gros (1814– ?), who used it to solve the (truly) ancient chinese ring puzzle (see [1, p. 51 ff]) or Frank Gray, who in 1953 introduced binary codes in which adjacent strings differ in a single bit only. Cummings [9] has interpreted g as a *coordinate sequence* of a Gray code. From Theorem 1 in that article it follows that g is even *strongly square-free*, i.e. $a := g$ does not contain a non-empty *abelian square*, which means, in our notation, there are no $m \in \mathbf{N}_0$ and $l \in \mathbf{N}$ such that

$$\forall k \in \{m+1, \ldots, m+l\} : a_{k+l} = a_{\sigma(k)} ,$$

σ being a permutation on $\{m+1, \ldots, m+l\}$. In fact, this follows easily from our proof of Theorem 0 as well: since a permutation can only affect even positions, we just have to remark that l must again be even, because otherwise $(d_{m+1}, \ldots, d_{m+l})$ and $(d_{m+l+1}, \ldots, d_{m+2l})$ would contain different numbers of 1s. □

1. A FINITE ALPHABET

Although g is, compared with the trivially square-free sequence $(k)_{k\in\mathbf{N}}$, economic in the sense that it uses smaller numbers for any finite part, it is unsatisfactory to depend on an *infinite* alphabet. Instead of considering d_μ in the Olive sequence, we now focus on (i_μ, j_μ), i.e. disregarding which disc is involved, we concentrate on the ways the discs are moving. Of these there are only six, namely

$$\alpha := (0,1),\ \beta := (1,2),\ \gamma := (2,0),\ \overline{\alpha} := (1,0),\ \overline{\beta} := (2,1),\ \overline{\gamma} := (0,2),$$

which will form the alphabet $\mathcal{A} := \{\alpha, \beta, \gamma, \overline{\alpha}, \overline{\beta}, \overline{\gamma}\}$. J.-P. Allouche *et al.* [2, Theorem 9] have shown that the sequence $c := (i_\mu, j_\mu)_{\mu\in\mathbf{N}}$ (named for N. Claus de Siam, who described the recursive solution in [7]) is square-free by recourse to the language of iterated morphisms. (For another interesting property of this sequence see Allouche and F. Dress [3].) We give a direct proof now, using only the following property of the TH itself:

LEMMA. *If* $\mu = 2^r(2k+1), r, k \in \mathbf{N}_0$, *then*

$$i_\mu = \{(1 + r \bmod 2)\, k\} \mod 3,$$
$$j_\mu = \{(1 + r \bmod 2)\,(k+1)\} \mod 3. \qquad \square$$

Proof. a) Let $n \in \mathbf{N}$ be such that $\mu < 2^n$ and put $i = 0, j = 2 - n \bmod 2$ in [13, Proposition 1]. Then, using $d_\mu = r + 1$, we get

$$\begin{aligned} i_\mu &= \{k(2 - n \bmod 2)\big((n - r - 1) \bmod 2 + 1\big)\} \mod 3 \\ &= \{(1 + r \bmod 2)\, k\} \mod 3, \end{aligned}$$

and similarly for j_μ.

b) As an alternative, we can prove this lemma directly by induction. Assume it is true for $1 \le \mu < 2^n$, $n \in \mathbf{N}_0$, when n discs move from peg 0 to peg $2 - n \bmod 2$. Then $\mu = 2^n$ is the move of disc $n+1$ from 0 to $1 + n \bmod 2$. For $2^n < \mu < 2^{n+1}$, discs 1 to n are transferred from $2 - n \bmod 2$ to $1 + n \bmod 2$; hence move μ is the same as move $\mu - 2^n$ with $0, 1, 2$ changed to $1, 2, 0$, respectively, if n is odd, and to $2, 0, 1$, if n is even. But then, since $\mu - 2^n$ is divisible by the same power of 2 as μ itself, we have $\mu - 2^n = 2^r\left(2(k - 2^{n-r-1}) + 1\right)$, and the formulas follow from $\left((1 + r \bmod 2)2^{n-r-1}\right) \bmod 3 = 2 - n \bmod 2$. $\square$

There are a couple of immediate consequences which we will need later:

COROLLARY.

o) $c_\mu \in \{\alpha, \beta, \gamma\} \Leftrightarrow r \bmod 2 = 0\,;$

i) $c_\mu \in \left(\{0, 1, 2\} \setminus \{(2\mu) \bmod 3\}\right)^2\,;$

ii) $c_\mu = \alpha, \beta, \gamma, \overline{\alpha}, \overline{\beta}, \overline{\gamma} \Leftrightarrow c_{2\mu} = \overline{\gamma}, \overline{\beta}, \overline{\alpha}, \gamma, \beta, \alpha,$ *respectively.* □

Proof. (o) is trivial; (i) and (ii) follow from

$$2\mu = (1 + r \bmod 2)\,(k + 2),$$

$$1 + (r + 1) \bmod 2 = 2(1 + r \bmod 2)\,,$$

both taken modulo 3, respectively. □

REMARK. Another direct consequence of the Lemma is (cf. [3, p. 10]):

$$\begin{aligned} &c_\mu = \alpha, \beta, \gamma, \overline{\alpha}, \overline{\beta}, \overline{\gamma} \Leftrightarrow \\ &\qquad \exists\, s, l \in \mathbf{N}_0 : \frac{\mu}{4^s} = 6l + 1, 6l + 3, 6l + 5, 12l + 10, 12l + 6, 12l + 2, \end{aligned}$$

respectively.

Our asymmetric choice of the first move being from 0 to 1 is here reflected in having, in some sense, twice as many unbarred as barred symbols in c, as remarked in [3, p.13]. □

Now we can prove the result of Allouche *et al.*:

THEOREM 1. *c is square-free.* □

Proof. Assume

$$\exists\, m \in \mathbf{N}_0\, \exists\, l \in \mathbf{N} \quad \forall \nu \in \{m + 1, \ldots, m + l\} : c_{\nu + l} = c_\nu\,.$$

If l is odd, then ν and $\nu + l$ have different parity. So every $\nu \in \{m + 1, \ldots, m + 2l\}$ has an even number of factors 2 by Corollary (o). Since of four consecutive numbers one has exactly one factor $2, l$ can only be 1. This, however, contradicts Corollary (i). Hence l must be even. But then, by virtue of Corollary (ii), the same argument as in the proof of Theorem 0 applies. □

2. SMALLER ALPHABETS

The six-letter alphabet $\mathcal{A}$ can still be reduced by building blocks of three moves. From Corollary (i) we learn that they must be of the form $(\tilde{\alpha}, \tilde{\gamma}, \tilde{\beta})$ with $\tilde{\xi} \in \{\xi, \overline{\xi}\}$. Only five of these actually do occur:

THEOREM 2. *Triples of elements of c form a square-free sequence h over the five-letter alphabet* $\{\mathrm{A}, \mathrm{B}, \Gamma, \Delta, \mathrm{E}\}$ *with*

$$\mathrm{A} := (\alpha, \overline{\gamma}, \beta), \ \ \mathrm{B} := (\alpha, \gamma, \overline{\beta}),$$

$$\Gamma := (\overline{\alpha}, \gamma, \beta), \ \ \Delta := (\alpha, \gamma, \beta), \ \ \mathrm{E} := (\overline{\alpha}, \gamma, \overline{\beta}). \qquad \square$$

Proof. From Corollary (o) we know: if γ occurs in c with a bar, its neighbors must be in odd positions and consequently unbarred. All the other triples turn up, the sequence starting

$$h = (\mathrm{A}, \mathrm{B}, \mathrm{A}, \Gamma, \mathrm{A}, \mathrm{B}, \Delta, \mathrm{E}, \mathrm{A}, \mathrm{B}, \mathrm{A}, \Gamma, \mathrm{A}, \mathrm{E}, \Delta, \Gamma, \mathrm{A}, \mathrm{B}, \mathrm{A}, \Gamma, \mathrm{A}, \mathrm{B}, \Delta, \mathrm{E}, \dots).$$

Clearly, h is square-free, since any square would lead to a square in c as well, contradicting Theorem 1. $\square$

REMARK. h (and consequently c) is *not* strongly square-free; can you spot an abelian square? (The existence of a strongly square-free string over a five-letter alphabet has been established by P.A.B. Pleasants [17, Theorem 2].) $\square$

Let me finally mention another instance of the TH to emerge as a microcosmos: it is known that the number of *states*, i.e. distributions of the discs among the three pegs, of the TH which can be reached from the initial state with all discs on peg 0, say, in and in no less than $\mu \in \mathbf{N}_0$ moves, is a power of 2, namely $2^{\beta(\mu)}$, where $\beta(\mu)$ is the number of non-zero bits of μ (see [13, Proposition 5]). ($2^{\beta(\mu)}$ also happens to be the number of odd entries in the μth row of Pascal's arithmetical triangle, as was realized by J. W. L. Glaisher [10, second § 14]; cf. [14, formula (4)].) Denoting $\beta(\mu) \bmod 2$ by m_μ, we obtain the *Thue-Morse sequence*

$$m := (0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, \dots),$$

which by the subsequent substitution of α for $(0, 1, 1)$, β for $(0, 1)$, and γ for (0) leads to the square-free sequence

$$t := (\alpha, \beta, \gamma, \alpha, \gamma, \beta, \alpha, \beta, \gamma, \beta, \alpha, \dots)$$

over the three-letter alphabet $\{\alpha, \beta, \gamma\}$. This is, of course, the smallest possible alphabet with an infinite square–free string (clearly, a square-free word over a two-letter alphabet will come to an end after three entries) with which the whole theory started in the work of Axel Thue [19, Satz 3], [20, Sätze 6, 7, 20].

Obviously, t (as in fact any word with more than 7 elements over a three-letter alphabet) is not strongly square-free. Maybe TH sequences hold a clue for a more direct approach to the question (cf. [6]), if there is an infinite strongly square-free string over a four-letter alphabet, which has been answered positively by V. Keränen [16] employing a computer-aided proof. (An abelian square of length $2 \cdot 6$ in h starts after position 6.)

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