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SQUARE-FREE TOWER OF HANOI SEQUENCES

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ABSTRACT. The sequence of moves in the optimal solution of the Tower of Hanoi with an arbitrary number of discs has recently been shown to lead to an example of an infinite square-free string over a six-letter alphabet by recourse to the theory of iterated morphisms. We present a direct approach to this result, using only properties of the Tower of Hanoi itself, which also reveals an implicit infinite square-free string with just five letters.

0. SQUARE-FREE STRINGS AND THE OLIVE SEQUENCE

Suppose you dispose of a reservoir of *letters* (or *symbols*), i.e. an at most countably infinite set $\mathcal{A}$, called an *alphabet*, and you are asked to construct a *word* (or *string*) of infinite length, i.e. a sequence $a \in \mathcal{A}^{\mathbf{N}}$, which does not contain any non-trivial immediate repetition, or *square*, i.e. there are no $m \in \mathbf{N}_0$ and $l \in \mathbf{N}$ such that

$$\forall k \in \{m+1, \ldots, m+l\} : a_{k+l} = a_k \,.$$

(For a concise survey on *square-free* words and their use in mathematics see [4].)

Assume $\mathcal{A} = \mathbf{N}$ and try an apparently economic approach, namely choose a_k as the smallest positive integer such that $(a_1, \ldots, a_k)$ does not contain any square. Then you come up with the following sequence:

$$(1,2,1,3,1,2,1,4,1,2,1,3,1,2,1,5,1,2,1,3,1,2,1,4,1,2,1,3,1,2,1,\ldots)$$

A connoisseur of the mathematical theory of the Tower of Hanoi (TH) puzzle (see [13] for a comprehensive survey) will immediately recognize the pattern of the disc numbers in the moves of the optimal solution to transfer a perfect tower of 5 discs from one peg to another. The, in fact unique, optimal solution of finite length $2^n - 1$ for the TH with an arbitrary number $n \in \mathbf{N}_0$ of discs has been excessively discussed in particular in the computer science literature — so excessively that an editor asked for "no more articles on this for a while".

On the other hand, apart from its playful appeal (cf. [11]), an object attracts the attention of mathematicians as soon as infinity is involved. A model of the TH with infinitly many discs was fundamental, for instance, in the discovery of the value of the average distance on the Sierpiński gasket (see [15] and [18]). So how do we define an infinite "optimal" sequence of moves for a TH with an inexhaustible provision of ever increasing discs, numbered $1, 2, \ldots$? (A rather bold assumption given that in the original description of the puzzle (see [7]) the discs were made of pure gold !) It can be done by recourse to one of the oldest observations about the finite optimal solution for n discs (see [8]), namely that the smallest disc 1 moves in every odd numbered move, always cyclically from peg 0 through peg 1 to peg 2, say, and that the even moves are then completely determined by the *divine rule* never to place a larger disc on a smaller one. This will take a tower of n discs from peg 0 to peg 1, if n is odd, and from peg 0 to peg 2, if n is even (cf. [13, Proposition 3]). We may therefore adopt this as a *definition* of the TH sequence or *Olive sequence* o (after its discoverer: Raoul Maurice Olive (1865– ?) was a nephew of the inventor of the TH, Édouard Lucas (cf. [14, Section 2], [12]), and at that time a student at the Lycée Charlemagne in Paris), as I prefer to call it to make a distinction from other sequences derived from the TH which we will encounter in the course of this note. The only ambiguity of this convention lies in fixing the direction the smallest disc moves in, and which will become apparent in some asymmetry of the results we will obtain. The Olive sequence is characterized, in fact overdetermined, by the triples $(d_\mu; i_\mu, j_\mu)$, $\mu \in \mathbf{N}$ being the move number, with $d_\mu \in \mathbf{N}$ the disc moving from $i_\mu \in \{0, 1, 2\}$ to $j_\mu \in \{0, 1, 2\}$.

Now the following is easy to prove, for instance by observing that the even moves form an Olive sequence too (cf. also [13, Proposition 1 (o)]) : if $\mu = 2^r(2k+1)$, with $r, k \in \mathbf{N}_0$, then $d_\mu = r + 1$, and $(d_\mu)_{\mu \in \mathbf{N}}$ is just the sequence with which we started our discussion. We will now show that it is square-free :

THEOREM 0. *The sequence* $g := (d_\mu)_{\mu\in\mathbf{N}}$ *of the discs moving in o represents a square-free string over the infinite alphabet* $\mathbf{N}$. □

Proof. Assume a square (of length $2l$) starts at position $m+1$, i.e.

$$\forall \nu \in \{m+1, \ldots, m+l\} : d_{\nu+l} = d_\nu ;$$

then l is necessarily even, since $d_\mu = 1$ if and only if μ is odd. So we may apply the general rule that $d_{2\mu} = d_\mu + 1$ to those of the νs which are in even positions and arrive at

$$\forall \nu \in \left\{\frac{m'}{2}+1, \ldots, \frac{m'}{2}+\frac{l}{2}\right\} : d_{\nu+l/2} = d_\nu$$

with m' either m or $m-1$, depending on the parity of m.

This yields a square of half the length of the original one, such that we will finally end up with a square for which $l=1$, a case we have already ruled out previously. □

REMARKS.

1. This proof is a nice example of the method of *infinite descent*, frequently employed by Fermat (cf. [5, p. 387]) and representing an early instance of the principle of *mathematical induction*.

2. The letter g denoting the sequence in Theorem 0 stands for either Louis Gros (1814– ?), who used it to solve the (truly) ancient chinese ring puzzle (see [1, p. 51 ff]) or Frank Gray, who in 1953 introduced binary codes in which adjacent strings differ in a single bit only. Cummings [9] has interpreted g as a *coordinate sequence* of a Gray code. From Theorem 1 in that article it follows that g is even *strongly square-free*, i.e. $a := g$ does not contain a non-empty *abelian square*, which means, in our notation, there are no $m \in \mathbf{N}_0$ and $l \in \mathbf{N}$ such that

$$\forall k \in \{m+1, \ldots, m+l\} : a_{k+l} = a_{\sigma(k)} ,$$

σ being a permutation on $\{m+1, \ldots, m+l\}$. In fact, this follows easily from our proof of Theorem 0 as well: since a permutation can only affect even positions, we just have to remark that l must again be even, because otherwise $(d_{m+1}, \ldots, d_{m+l})$ and $(d_{m+l+1}, \ldots, d_{m+2l})$ would contain different numbers of 1s. □