1. Finsler metric from the contact geometrical viewpoint

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seen that the statement of Theorem 0.1 holds even without the assumption $[\gamma''(t), \gamma'''(t)] \neq 0$.

The 4-vertex theorem in the Minkowski plane is by no means new. An equivalent statement can be found in [Bl 1]; see also [Ge, Gu 1,2, He 1,2, Su] (note another term, the *relative differential geometry*, classically used to describe the situation).

The point of view in this paper is that of contact geometry which, I believe, clarifies the matter and makes it possible to extend naturally many familiar results from the Euclidean setting to the more general Minkowski and Finsler ones. For an approach to the 4-vertex theorem and related results as theorems of symplectic and contact topology see, e.g., [A 1, A 4].

1. Finsler metric from the contact geometrical viewpoint

Finsler geometry describes the propagation of light in an inhomogeneous anisotropic medium. This means that the velocity of light depends on the point and the direction. There are two equivalent descriptions of this process corresponding to the Lagrangian and the Hamiltonian approaches in classical mechanics.

On the one hand, one may study the rays of light, that is, the shortest paths between points. The optical properties of a medium are described by a strictly convex smooth hypersurface, called the *indicatrix*, in the tangent space at each point. The indicatrix consists of the velocity vectors of the propagation of light at a point in all directions. It plays the role of the unit sphere in Riemannian geometry.

The distance $d(x,y)$ between points x and y is the least time it takes light to travel from x to y. If the indicatrices are not centrally symmetric this distance may not be symmetric: $d(x,y) \neq d(y,x)$. However it still satisfies the triangle inequality:

$$d(x,y) + d(y,z) \geq d(x,z).$$

Minkowski geometry is a particular case of Finsler geometry in affine space in which the indicatrices of all points are identified by parallel translations. The rays of light in Minkowski geometry are straight lines.

On the other hand, one may study the wave fronts. The wave front of a point is the hypersurface that consists of points which light can reach from the given point in a fixed time. A wave front is characterized by its contact elements (hyperplanes in the tangent spaces at the points of the front tangent to it) cooriented by the direction of the time evolution of the front. This evolution is described by a vector field in the space of all cooriented contact elements.

We recall in this section (without proofs) the relevant facts from symplectic and contact geometry — see [A 2, A 3].

Let M^n be a smooth manifold and $\pi : T^*M \to M$ its cotangent bundle. When needs be one introduces local coordinates in T^*M,

$$(q,p) = (q_1, \dots, q_n, p_1, \dots, p_n),$$

where q are position coordinates in M and p are the corresponding momenta coordinates in the fibers of the cotangent bundle. Denote by λ_0 the Liouville differential 1-form on T^*M. The value of λ_0 on a tangent vector v to T^*M at point (x,θ), where $x \in M, \theta \in T^*_xM$, is, by definition, $\theta\big(d\pi(v)\big)$. In coordinates, $\lambda_0 = p\,dq$ $(= \sum p_i\,dq_i)$. The 2-form $d\lambda_0$ is the canonical symplectic form in T^*M.

The space of cooriented contact elements is the spherization ST^*M of the cotangent bundle. Consider the principle $\mathbf{R}^*_+$ - bundle

$$p : T^*M - M \to ST^*M$$

$(T^*M - M$ is the complement to the zero section); its fiber over a cooriented contact element consists of the linear functionals vanishing on this contact element and positive on its positive side. The codimension 1 distribution $\operatorname{Ker}\lambda_0$ on $T^*M - M$ projects to the canonical contact structure in ST^*M.

A Finsler metric on M is determined by a (Hamiltonian) function H on T^*M. This function satisfies the following assumptions:

1) H is a nonnegative function, homogeneous of degree 1 in momenta, i.e., $H(q,tp) = tH(q,p)$ for all $t > 0$;
2) The level hypersurface $S = H^{-1}(1)$ is fiberwise star-shaped, i.e., each intersection $S_x = S \cap T^*_xM$, $x \in M$, transversely intersects every ray from the origin in the linear space T^*_xM.
3) The level hypersurface $S = H^{-1}(1)$ is fiberwise quadratically convex, i.e., each intersection $S_x = S \cap T^*_xM$, $x \in M$, is quadratically convex in the linear space T^*_xM.

The surface S_x is sometimes called the *figuratrix*. It may be thought of as the set of "unit covectors" in T^*_xM.

Denote by ξ the Hamiltonian vector field of H, that is the field such that $i_\xi d\lambda_0 = dH$. In local coordinates,

$$\xi = H_p\,\partial/\partial q - H_q\,\partial/\partial p.$$

The field ξ is tangent to the hypersurface S. Let ϕ_t be the time-t map of the flow ξ. Denote by λ the restriction of the Liouville form to S.

THEOREM 1.1. *Let the Hamiltonian function satisfy the above conditions 1)–2). Then:*

a) *The form λ is a contact form, that is, $\lambda \wedge (d\lambda)^{n-1} \neq 0$ everywhere on S.*

b) *The field ξ is the characteristic vector field of the form λ, that is, $i_\xi d\lambda = 0$, $\lambda(\xi) = 1$, and the flow ϕ_t preserves the form λ for all t.*

The hypersurface S being fiberwise star-shaped, it is identified with ST^*M, and the contact form λ determines the canonical contact structure in ST^*M. Conversely, a contact form λ for the canonical contact structure in ST^*M is a section ϕ of the bundle $p : T^*M - M \to ST^*M$ such that $\phi^*\lambda_0 = \lambda$. The image of this section is a fiberwise star-shaped hypersurface $S \subset T^*M$, and one can reconstruct the homogeneous Hamilton function H by $S = H^{-1}(1)$.

The one-parameter group ϕ_t describes the time evolution of cooriented contact elements of M mentioned at the beginning of the section. This flow will be referred to as the *geodesic flow* in the space of cooriented contact elements.

EXAMPLE. Let M be a Riemannian manifold and $H(q,p) = |p|$. Then ξ is the usual geodesic flow: each coorented contact element moves with the unit speed in its positive normal direction.

We assume that the figuratrices S_x are quadratically convex. The indicatrix I_x at point $x \in M$ consists of the velocity vectors of the foot points of the contact elements in S_x under the flow ξ. That is,

$$I_x = \{d\pi\big(\xi(x,\theta)\big), \quad \theta \in S_x \subset T_x^*M\}.$$

DEFINITION. Let X be a smooth strictly convex star-shaped hypersurface in a vector space V. For every $x \in X$ there exists a unique functional $y \in V^*$ such that $y(x) = 1$ and Ker $y = T_xX$. The set of such functionals for all $x \in X$ is called the *dual hypersurface* and is denoted by X^*.

Note that X^* is strictly convex and star-shaped too; note also that $(X^*)^* = X$.

THEOREM 1.2. *The indicatrix I_x and the figuratrix S_x are dual to each other for every $x \in M$.*

To the field of indicatrices there corresponds a (Lagrangian) function L on the tangent bundle TM: this function is homogeneous of degree 1 in tangent vectors, and $L^{-1}(1) \cap T_xM = I_x$ for all $x \in M$. This function gives the length of a tangent vector in Finsler geometry. Trajectories of light in Finsler geometry are the extremals of the functional $\int L(q, \dot{q})\, dt$.

THEOREM 1.3. *These extremals are the projections to M of the trajectories of the vector field ξ.*

Thus the Hamiltonian vector field ξ of the Hamiltonian function H describes the propagation of light in an inhomogeneous anisotropic medium. In the case of Minkowski geometry H depends on the momenta variables only. The trajectories of light in Minkowski geometry are straight lines, and the indicatrix is identified with the time-1 front of the origin. The cooriented contact elements of this front are the time-1 images in the geodesic flow of all contact elements at the origin.

Let $N \subset \mathbf{R}^n$ be a cooriented hypersurface in Minkowski space. The geodesic flow trajectories of the foot points of the cooriented contact elements of N will be called (Minkowski) *normals* of N. Note that the normals may change if the coorientation of N is reversed. The reader interested in differential geometry of Finsler manifolds is referred to [Ru], and to [Bu] for the case of Minkowski geometry.

2. Minkowski geometry associated with a parametrized curve

Return to the situation of the Introduction: $\gamma(t)$ is a smooth closed strictly convex parametrized plane curve satisfying the condition $[\gamma''(t), \gamma'''(t)] \neq 0$ for all t. The lines $l(t)$ generated by the acceleration vectors $\gamma''(t)$ constitute a smooth transverse line field along $\gamma(t)$. The condition $[\gamma''(t), \gamma'''(t)] \neq 0$ ensures that infinitesimally close lines from the family $l(t)$ intersect, therefore their envelope is bounded.

Give γ the inward coorientation. Then γ determines a curve $\tilde{\gamma}$ in the space of cooriented contact elements of the plane. The curve $\tilde{\gamma}$ is Legendrian, that is, tangent to the contact structure in the space of cooriented contact elements.

THEOREM 2.1. *There exists a unique, up to a multiplicative constant, Minkowski metric in the plane such that the lines $l(t)$ are the Minkowski normals of the cooriented curve γ.*