# 4. HOMOMORPHISMS OF SURFACE GROUPS


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The following is an interesting illustration of Theorem 3.31.

3.33. EXAMPLE: SELF-MAPS OF THE REAL PROJECTIVE PLANE. By considering the Puppe exact sequence [12, p. 238], [13, p. 3] associated to a map $S^1 \to S^1$ of degree 2, one finds that each pointed homotopy class of maps from a real projective plane to a real projective plane is determined by its degree, and the possible values are 0, 2, and the odd positive integers. In particular, the same holds for the (unpointed) homotopy classes of maps.

A degree zero map is given by collapsing the source surface to a point. This is of type (a).

A degree two map is given by collapsing an unorientable loop to a point to obtain a two-sphere, and then composing with a double covering of the projective plane. This is an orientation-false pinching composed with a double covering, so is of type (b).

An odd positive integer degree map is given by taking an odd positive degree covering of one Möbius band by another, and then collapsing the boundaries to points. This is a branched covering with a single branch point, so is of type (c).

In the usual way, the homotopy classes of self-maps of the real projective plane form a monoid under composition; to calculate composites one need only calculate the degree, and that can be done easily, even using the algorithm given here. Thus we can identify each homotopy class with its degree, and examine the binary operation induced by composition. We find that the monoid is obtained from the usual multiplicative monoid of non-negative integers by identifying two distinct non-negative integers if and only if they are even and are equal modulo 4.

## 4. HOMOMORPHISMS OF SURFACE GROUPS

Throughout this section, let $\alpha\colon G_1 \to G_2$ be a homomorphism of infinite surface groups, and $G_1 = \langle S_1 \mid r_1 \rangle$, $G_2 = \langle S_2 \mid r_2 \rangle$ be surface group presentations.

4.1. REVIEW. The arguments of Sections 2, 3 give us a method for finding a normal form for $\alpha$, and hence for calculating the degree of $\alpha$.

Let us itemize the steps performed.

We choose a homomorphism of free groups $A\colon \langle S_1 \mid \ \rangle \to \langle S_2 \mid \ \rangle$ which induces $\alpha$, and we choose, for some non-negative integer $d$, elements $w_1, \dots, w_d$ of $\langle S_2 \mid \ \rangle$, and elements $\epsilon_1, \dots, \epsilon_d$ in $\{1, -1\}$, such that

$$(4.1) \qquad A(r_1) = \prod_{i=1}^{d} w_i \, r_2^{\epsilon_i} \, w_i^{-1} \text{ in } \langle S_2 \mid \ \rangle .$$

We then use $A$ and (4.1) in Construction 2.10 to construct a cellular map $\beta\colon X_1 \to X_2$ realizing $\alpha$. Here $S_1$ (resp. $S_2$) is identified with a basis of the free fundamental group of a specified subgraph of $X_1$ (resp. the whole one-skeleton of $X_2$) with a specified base vertex; also, $r_1$ (resp. $r_2$) corresponds to the boundary cycle of a certain subdivided face (resp. the unique face). By construction, $\beta$ restricts to a graph morphism between the specified subgraphs, and the resulting homomorphism of free fundamental groups agrees with $A$.

We apply the algorithm of Section 3 to $\beta$ to obtain a new cellular map $\beta'\colon X_1' \to X_2'$ which is in the normal form given by Theorem 3.31. Here $X_2'$ is obtained from $X_2$ through Constructions 3.1 and 3.30, and there is a natural map from the one-skeleton of $X_2'$ to the one-skeleton of $X_2$, and both complexes have natural base vertices, and both base vertices will be denoted $v_2$. By Remarks 2.5, we can trace through the steps of the algorithm and for each transform of $X_1$, we can identify $S_1$ with a basis of the fundamental group of a subgraph with a base vertex. Thus for any set $E'$ of edges which corresponds to a maximal subtree of the one-skeleton of the dual complex of $X_1'$, we can identify $S_1$ with a basis of the fundamental group of the one-skeleton of $X_1' - E'$. Throughout the algorithm $S_1$ is altered only up to homotopy and change of base vertex. Moreover, up to conjugacy and inverse, $r_1$ agrees with the boundary cycle of the resulting subdivided large face expressed in terms of the basis $S_1$. Now $\beta'$ gives a new lifting $A'$ of $\alpha$, via the labelling. The boundary label of each face in $X_1'$ corresponds to a conjugate of $r_2^{\pm 1}$, and the subdivided large face gives a description of $A'(r_1)$ as a product of conjugates of $r_2^{\pm 1}$, by viewing the large face as a compressed version of Figure 2.4. Now we get an expression

$$(4.2) \qquad A'(r_1) = \prod_{i=1}^{d'} w_i' \, r_2^{\epsilon_i'} \, w_i'^{-1}$$

in which $d'$ is the number of $F$-faces, and hence equals the degree of $\alpha$.

For some purposes, it is convenient to have a new generating set $S_1'$ of $G_1$ adapted to the normal-form map. This can be thought of as a change of basis within the free group, but we prefer to think of it as giving a new free group mapping onto $G_1$, with a specified isomorphism to the old free group, with the property that the new relator $r_1'$ arising from the boundary cycle of the subdivided face corresponds to $r_1$, up to conjugacy and inverse.

To define $S_1'$, we first choose a set of edges to erase in $X_1'$ as follows. Choose a maximal forest of $E$-edge-adjacent faces in $X_1$, and erase the $E$-edges, and then choose a maximal tree of $V$-edge-adjacent faces and erase the $V$-edges. It is clear from Figure 3.1 that, in the prepinching regions, the interior $E$-edges get erased, and the interior $V$-edges do not. In the one-skeleton of the resulting CW-surface, choose a base vertex $v_1$ which maps to $v_2$, and choose a maximal tree, and collapse the edges; notice that these are all $E$-edges, since the $V$-edges are loops. This gives us a surface group presentation, $\pi(X_1', v_1) = \langle S_1' \mid r_1' \rangle$. Now $\pi(\beta'^{(1)}, v_1)\colon \pi(X_1'^{(1)}, v_1) \to \pi(X_2'^{(1)}, v_2)$ determines a homomorphism $A''\colon \langle S_1' \mid \ \rangle \to \langle S_2 \mid \ \rangle$ of free groups, and we get an equation

$$A''(r_1') = \prod_{i=1}^{d''} w_i'' r_2^{\epsilon_i''} w_i''^{-1} \tag{4.3}$$

closely related to the normal form, in which $d''$ is the degree of $\alpha$.

Here all unerased $V$-loops, which include all the $V$-loops occurring in prepinchings, determine elements of $S_1'$ which are sent to 1 under $A'$. Thus the algorithm gives us a distinguished set $K \subseteq S_1'$ of generators which go to 1.

We now want to examine in detail what can be said in each of the three types of normal norm.

4.2. THE DEGREE ZERO CASE. Suppose case (a) of Theorem 3.31 holds.

Here $r_2$ loses its significance, and we are studying a homomorphism from a surface group to the free group $\pi(X_2^{(1)}, v_2) = \langle S_2 \mid \ \rangle$.

Form a labelled graph $\Gamma$ by collapsing each $E$-sphere to an edge. The labelling immerses $\Gamma$ in the graph $X_2^{(1)}$, since no two $E$-spheres at a vertex have the same $E$-label. In particular, if the induced map of fundamental groups $\pi(\beta, v_1)\colon \pi(X_1, v_1) \to \pi(X_2^{(1)}, v_2)$ is surjective, then the labelling identifies $\Gamma = X_2^{(1)}$.

Our erasing procedure erases all but one $E$-edge in each punctured $E$-sphere, and then erases $V$-loops incident to distinct faces as often as possible, leaving a single face. The one-skeleton is then a copy of $\Gamma$ with bouquets of $V$-loops at each vertex. We then collapse a maximal subtree of $\Gamma$ to a vertex, to obtain the surface group presentation $G_1 = \langle S_1' \mid r_1' \rangle$. Every element of $S_1'$ is either an edge of the collapsed $\Gamma$, or is an unerased $V$-loop. Recall that $K$ denotes the set of elements of $S_1'$ corresponding to unerased $V$-loops. Then the complement, $S_1' - K$, is in bijective correspondence with the edge set of the collapsed $\Gamma$. If we were to collapse the unerased $V$-loops to vertices, we would have a face with boundary label a relation in the fundamental groupoid of $\Gamma$, but this is a free groupoid, so the relation represents a trivial element. That is, $r_1'$ lies in the normal closure of $K \subset S_1'$.

This proves that any surjective homomorphism from a surface group to a free group can be expressed in the form $\langle S_1' \mid r_1' \rangle \to \langle S_1' \mid r_1', K \rangle$ where $K$ is a subset of $S_1'$ whose normal closure contains $r_1'$.

One can extract even more information from the diagram. For example, it is natural to divide in half all those edges of $\Gamma$ which lie outside the maximal subtree, and subdivide the edges and faces of $X_1'$ which map to these. This introduces an orientable $V$-loop around the equator of certain punctured $E$-spheres, and we can erase one old $V$-loop for each equator we add. The surface obtained by deleting these equators from $X_1'$ maps to the subtree of $\Gamma$ obtained by deleting a point from each edge outside the maximal subtree. Hence we have a punctured subsurface which maps to a tree, so its fundamental group is collapsed. The surface $X_1'$ can be recovered from the punctured surface by identifying boundary components in pairs. The effect on the fundamental group is to form an HNN-extension which adds a new generator conjugating one of the boundary components to the other, and the new generator corresponds to one of the specified generators of the fundamental group of $\Gamma$. This can be used to give quite a precise normal form, but we are still some distance from recovering all the information that is currently known. Zieschang [17, Satz 2] showed that any surjective homomorphism from an orientable surface group onto a free group can be expressed in the form

$$\langle x_1, y_1, \ldots, x_n, y_n \mid (x_1, y_1) \cdots (x_n, y_n) \rangle$$
$$\to \langle x_1, y_1, \ldots, x_n, y_n \mid (x_1, y_1) \cdots (x_n, y_n), x_1, \ldots, x_n, y_{r+1}, \ldots, y_n \rangle,$$

where $0 \leq r \leq n$. Grigorchuk and Kurchanov [7, Theorem 1] showed that any surjective homomorphism from an unorientable surface group onto a free group can be expressed in exactly one of two forms

$$\langle z_1, z_2, \ldots, z_n \mid z_1^2 z_2^2 \cdots z_n^2 \rangle \\ \to \langle z_1, z_2, \ldots, z_r \mid z_1^2 z_2^2 \cdots z_r^2, z_1 z_2, z_3 z_4, \ldots, z_{2r-1} z_{2r} \rangle ,$$

where $z_{2r+1}, \ldots, z_n$ are either all sent to $z_{2r}$, where $n$ is even and $0 < 2r \leq n$, or all sent to 1, where $0 \leq 2r < n$. An elegant proof can be found in [8].

Ol'shanskii [15, Section 2] used diagram techniques similar to those used here to obtain some of the above results independently.

It is interesting to note that $V$-loops frequently occur in the literature. Edmonds [3] and Skora [16], in the course of their arguments, find it necessary to prove that, for any surface map of degree zero, there exists a non-separating $V$-loop; Skora uses a non-separating point of the graph $\Gamma$, except in the case where $\Gamma$ is a tree and the map is trivial. Ol'shanskii's arguments for maps from surface groups to free groups are based on proving that there exists a non-collapsable $V$-loop. Gabai [4] used three-dimensional topology to show that every non-injective homomorphism between surface groups can be represented by a diagram with a non-collapsable $V$-loop.

We now turn to the nonzero degree case, and describe the group-theoretic formulation of branched covers.

4.3. THE BRANCHED COVERING CASE. Consider any non-negative integers $n$, $m$, $p$, with $m = 0$ or $n = 0$, and positive integers $d_1, \ldots, d_p$. Let

$$G = \langle x_1, y_1, \ldots, x_n, y_n, z_1, \ldots, z_m, t_1, \ldots, t_p \\ \mid (x_1, y_1) \cdots (x_n, y_n) z_1^2 \cdots z_m^2 t_1 \cdots t_p, t_1^{d_1}, \ldots, t_p^{d_p} \rangle .$$

There is a canonical map from $G$ to the surface group

$$G_2 = \langle x_1, y_1, \ldots, x_n, y_n, z_1, \ldots z_m \mid (x_1, y_1) \cdots (x_n, y_n) z_1^2 \cdots z_m^2 \rangle ,$$

obtained by annihilating the $t_k$.

The *Euler characteristic* of $G$ is defined as

$$\chi(G) = 2 - m - 2n - p + \sum_{i=1}^{p} \frac{1}{d_i} .$$

For example, if $p = 0$ then $G = G_2$, and here the Euler characteristic plus the rank equals 2, where the *rank* is the minimum number of generators, or equivalently, the size of the generating set in the surface group presentation.

It is known that $G$ acts, with compact quotient, as a group of isometries on a sphere, plane or hyperbolic disc, depending as $\chi(G)$ is positive, zero, or negative, respectively. Any subgroup $H$ of finite index is again of this form, and the Riemann-Hurwitz formula says that $\chi(H) = (G : H)\chi(G)$.

If we choose a surface subgroup $G_1$ in $G$ of finite index, then we get a homomorphism of surface groups $G_1 \to G_2$. A homomorphism arising in this way is called a *branched covering homomorphism of surface groups*. It is not difficult to construct the corresponding cellular map of CW-surfaces in normal form, and find that it is a $d$-fold branched covering, where $d = (G : G_1)$, and the $d_1, \dots, d_p$ can be taken as the branching degrees. Conversely, any cellular map of CW-surfaces which is a branched covering has an associated group homomorphism of this form.

There is an *orientation map* from $G$ to $\{\pm 1\}$ which sends the $x_i$, $y_i$, $t_i$ to 1, and the $z_i$ to $-1$. It follows that branched covering homomorphisms of surface groups are orientation-true, so for infinite surface groups, the value of $\left|\sum_{i=1}^{d} \epsilon_i\, \epsilon(w_i)\right|$ in (4.1) is independent of the lifting chosen.

Let us take presentations and diagrams corresponding to the branched covering. Consider an edge $e$ in $E$, and a distinguished occurrence of $e$ in the boundary cycle of the single face $f$ in $F$, and two distinct $e$-adjacent $f$-faces, denoted $f_i$, $f_j$. These have associated a $w_i$ and a $w_j$ representing paths back to the base vertex, so $w_i^{-1}w_j$ represents a path between the base vertices of $f_i$ and $f_j$, and for the purposes of checking signs, we may assume the base vertex is incident to $e$. Since the $f$-faces are well $e$-joined, $\epsilon(w_i^{-1}w_j)$ describes whether the two (distinguished) $e$-edges in the two $f$-faces would be identified with a twist, or not, that is, have the same, or different, signs, respectively, in the two occurrences in the boundary cycle of $f$. But $\epsilon_i^{-1}\epsilon_j$ describes whether the two adjacent $f$-faces have the same orientation of not. Thus $\epsilon(w_i^{-1}w_j) = \epsilon_i^{-1}\epsilon_j$. Hence, for this choice of presentation, $\left|\sum_{i=1}^{d} \epsilon_i\, \epsilon(w_i)\right| = d = (G : G_1)$. In summary, the degree of a branched homomorphism of infinite surface groups is given by $(G : G_1)$.

Let $N$ denote the kernel of $G \to G_2$. Then

$$(G : G_1) \geq (G : G_1N) = (G/N : G_1N/N) = (G_2 : \operatorname{Im} G_1),$$

so the degree is at least $(G_2 : \operatorname{Im} G_1)$.

Notice that the degree is exactly $(G_2 : \operatorname{Im} G_1)$ if and only if $G_1 = G_1 N$, that is, $N$ lies in $G_1$. But $N$ is generated by torsion elements, and $G_1$ is torsion-free, so the latter holds if and only if $N$ is trivial, that is, $G = G_2$, which is the case $p = 0$. Here we simply have an inclusion of finite index, which corresponds to an (unbranched) covering.

In case (b) of Theorem 3.31, adding the relations to $\pi(X_1', v_1) = G_1$ which annihilate the pinched generators leaves a surface group presentation. Let us invent terminology to express this.

4.4. THE PINCHING CASE. A *pinching homomorphism of surface groups* is a homomorphism which can be put in the form $\langle S \mid r\rangle \to \langle S \mid r, K\rangle$ where $\langle S \mid r\rangle$ is a surface group presentation, and $K$ is a subset of $S$ such that deleting the occurrences of elements of $K$ from $r$ leaves a word $r'$ such that $\langle S - K \mid r'\rangle$ is a surface group presentation. Notice that the parity of the homomorphism is odd. If some element of $K$ occurs twice with the same sign in $r$, then the homomorphism is orientation-false, and otherwise it is orientation-true.

It can be shown that a pinching homomorphism of surface groups can be uniquely expressed in the form $\langle S \mid r\rangle \to \langle S \mid r, K\rangle$ where $K \subseteq S$ and exactly one of the following holds:

$$S = \{x_1, y_1, \dots, x_n, y_n\},\ K = \{x_1, y_1, \dots, x_m, y_m\},$$
$$\text{and } r = (x_1, y_1)\cdots(x_n, y_n), \text{ where } 0 \le m \le n;$$
$$S = \{x_1, y_1, \dots, x_n, y_n, z_1, \dots, z_m\},\ K = \{x_1, y_1, \dots, x_n, y_n\},$$
$$\text{and } r = (x_1, y_1)\cdots(x_n, y_n)z_1^2\cdots z_m^2, \text{ where } 0 \le m,\ 1 \le n;$$
$$S = \{x_1, y_1, \dots, x_n, y_n, z_1, \dots, z_m\},\ K = \{z_1, \dots, z_m\},$$
$$\text{and } r = (x_1, y_1)\cdots(x_n, y_n)z_1^2\cdots z_m^2, \text{ where } 0 \le m,\ 1 \le n;$$
$$S = \{z_1, \dots, z_m\},\ K = \{z_1, \dots, z_n\}$$
$$\text{and } r = z_1^2\cdots z_m^2 \text{ where } 0 \le n < m.$$

The first two types are orientation-true, and the last two types are orientation-false.

Suppose now that $\alpha\colon G_1 \to G_2$ factors as a pinching homomorphism of surface groups $\alpha'\colon G_1 \to \operatorname{Im}\alpha$, followed by an inclusion of finite index $\alpha''\colon \operatorname{Im}\alpha \to G_2$. We wish to verify that $\mathcal{G}(\alpha) = (G_2 : \operatorname{Im}\alpha)$.

It is straightforward to construct a lifting $A$ and an equation (4.1) with $d = (G_2 : \operatorname{Im}\alpha)$, so we have $\mathcal{G}(\alpha) \le (G_2 : \operatorname{Im}\alpha)$, and we may assume $\mathcal{G}(\alpha) < (G_2 : \operatorname{Im}\alpha)$. We wish to obtain a contradiction.

Notice that the parity of $\alpha'$ is odd, so $\alpha'$ does not factor through a free group, and hence $\alpha$ itself cannot factor through a free group. Thus $\mathcal{G}(\alpha) > 0$.

Let $d = \mathcal{G}(\alpha)$. We may assume that we started with a lifting $A$, and an equation (4.1), that is, $d$ is smallest possible. Thus in the process of applying the algorithm of Section 3, we perform no cancellation of $F$-faces, and we finish with a Normal Form map of degree $d$. We are not in case (a), since $d > 0$, and we are not in case (b) or (c), since $d < (G_2 : \operatorname{Im}\alpha)$. This is impossible, as desired.

Thus we have proved the following.

4.5. THEOREM (Kneser-Edmonds-Skora). *If $\alpha\colon G_1 \to G_2$ is a homomorphism between infinite surface groups, then exactly one of the following holds.*

(a) *The homomorphism $\alpha$ factors through a surjective homomorphism from $G_1$ to a free group; here $\mathcal{G}(\alpha) = 0 < (G_2 : \operatorname{Im}\alpha)$.*

(b) *For some positive integer $d$, $\alpha$ factors as a pinching homomorphism followed by an index $d$ inclusion; here $\mathcal{G}(\alpha) = d = (G_2 : \operatorname{Im}\alpha)$.*

(c) *For some positive integer $d$, $\alpha$ is a non-injective $d$-fold branched covering homomorphism of surface groups; here $\mathcal{G}(\alpha) = d > (G_2 : \operatorname{Im}\alpha)$.* □

Notice that in type (b) we have the usual factorization as a surjection followed by a (finite index) inclusion, while in type (c) we have a rather unusual finite index inclusion followed by a surjection. In type (a), we have a special surjection to a free group, with kernel generated by at least half the generators in a suitable surface group presentation, followed by a homomorphism which need not be injective.

4.6. COROLLARY (Kneser [10], [11]). *If a homomorphism between infinite surface groups has degree* 1 *then it is a (possibly bijective) pinching homomorphism.* □

4.7. COROLLARY. *If $G$ is a surface group with negative Euler characteristic, and $\alpha$ is an endomorphism of $G$, then either $\alpha$ is an automorphism, or the image of $\alpha$ has infinite index in $G$, and the kernel of $\alpha$ is generated as normal subgroup by a set consisting of at least half the generators in some surface group presentation of $G$.*

*Proof.* If $(G : \operatorname{Im}\alpha)$ is infinite, then $\operatorname{Im}\alpha$ is a free group, and by a Grigorchuk-Kurchanov-Zieschang result recovered in Case 4.2, the kernel of $\alpha$ is generated as normal subgroup by a set consisting of at least half the generators in some surface group presentation of $G$.

This leaves the case where $\operatorname{Im}\alpha$ has finite index $n$ in $G$. To see that $n = 1$, we suppose that $n > 1$ and obtain a contradiction as follows. By the Riemann-Hurwitz formula, and the fact that $\chi(G) < 0$, we see that $\chi(\operatorname{Im}\alpha) = n\chi(G) > \chi(G)$, so the rank of $\operatorname{Im}\alpha$ is less than the rank of $G$. This is impossible, since $\operatorname{Im}\alpha$ is a quotient of $G$, so $n = 1$. Hence $\alpha$ is surjective.

Since $\alpha$ cannot factor through a group of rank strictly smaller than that of $G$, we see that $\alpha$ cannot factor through a non-trivial pinching homomorphism. By Theorem 4.5, we see that $\alpha$ is a branched covering homomorphism. Thus $G$ has finite index $m$ in some group

$$H = \langle x_1, y_1, \dots, x_n, y_n, z_1, \dots z_m, t_1, \dots, t_p \mid (x_1, y_1)\cdots(x_n, y_n)z_1^2 \cdots z_m^2 t_1 \cdots t_p, t_1^{d_1}, \dots, t_p^{d_p} \rangle,$$

where $G = \langle x_1, y_1, \dots, x_n, y_n, z_1, \dots z_m \mid (x_1, y_1)\cdots(x_n, y_n)z_1^2 \cdots z_m^2 \rangle$. By the Riemann-Hurwitz formula, and the fact that $\chi(G) < 0$, we see that $\chi(G) = m\chi(H) \le \chi(H)$, so $0 \ge \chi(G) - \chi(H) = p - \sum_{i=1}^{p} \frac{1}{d_i} \ge 0$. It follows that $m = 1$, and that $\alpha$ is bijective. □

We can also recover Kneser's description of degree.

4.8. THEOREM (Kneser [10], [11]). *Let $\alpha\colon G_1 \to G_2$ be a homomorphism of infinite surface groups, and consider an equation* (4.1) *arising from some lifting of $\alpha$.*

(i) *If $\alpha$ is orientation-true, then $\mathcal{G}(\alpha) = \left|\sum_{i=1}^{d} \epsilon_i\, \epsilon(w_i)\right|$, where the map $\epsilon\colon \langle S_2 \mid \ \rangle \to \{\pm 1\}$ is induced from the orientation map of $G_2$.*

(ii) *If $\alpha$ is orientation-false, and either $d$ is even, or the index $(G_2 : \operatorname{Im}\alpha)$ is infinite, then $\mathcal{G}(\alpha) = 0$.*

(iii) *If $\alpha$ is orientation-false, and $d$ is odd, and $(G_2 : \operatorname{Im}\alpha)$ is finite, then $\mathcal{G}(\alpha) = (G_2 : \operatorname{Im}\alpha)$.*

*Moreover, the lifting $A$ can be chosen so that $d = \mathcal{G}(\alpha)$, with the original choice of presentations.* □

This result can be used to prove a theorem of Nielsen's which predates Kneser's result.

4.9. THEOREM (Nielsen [14, Section 26], [9]). *If $\langle S \mid r\rangle$ is a surface group presentation of a (surface) group $G$, and $\alpha$ is an automorphism of $G$, then there exists an automorphism $A$ of the free group $\langle S \mid \ \rangle$ which maps $r$ to a conjugate of $r$ or $r^{-1}$, such that the induced map on $G$ is $\alpha$.*

*Proof.* This is clear if $G$ is finite, so we may assume that $G$ is infinite. Since $\alpha$ is an automorphism, the kernel is trivial, so $\alpha$ does not have degree zero, and no pinching takes place. Thus $\alpha$ must be a branched covering homomorphism, by Theorem 4.5. We saw in Definition 4.3 that branched covering homomorphisms are orientation-true. It follows from Theorem 4.8 (a), that, among orientation-true maps, the degree is multiplicative with respect to composition. Thus $\mathcal{G}(\alpha)\mathcal{G}(\alpha^{-1}) = \mathcal{G}(1) = 1$. Thus $\mathcal{G}(\alpha) = 1$. By the final part of Theorem 4.8, we can choose a lifting of $\alpha$ to an endomorphism $A$ of the free group on $S$ which sends $r$ to a conjugate of $r$ or $r^{-1}$. A theorem of Zieschang [17] then shows that $A$ is an automorphism. (A simple proof of surjectivity, using Fox derivatives, is given in Theorem V.4.11 of [1], and injectivity is proved using Nielsen reductions, as in Theorem I.10.5 of [1].) □

The foregoing argument contains elements of the original proof by Nielsen, and of the algebraic proof by Zieschang [17], [18, Corollary 5.4.3].

4.10. REMARKS. Recall that for two groups $G_1$ and $G_2$, the set of group homomorphisms from $G_1$ to $G_2$ is partitioned into orbits under the natural action of the group $\mathrm{Aut}(G_1)$ via composition. Two homomorphisms in the same orbit are said to be *strongly equivalent*.

Without going into details, let us describe some known results.

Case 4.2, above, mentions surjective homomorphisms from surface groups to free groups. Such homomorphisms have been thoroughly analyzed by algebraic techniques, starting with the work of Zieschang [17], and Ol'shanskii [15], and culminating in the work of Grigorchuk and Kurchanov [7]. This work is distilled in [8] where it is shown that if $\alpha_1, \alpha_2 \colon G_1 \to G_2$ are homomorphisms from a surface group to a free group, then they are strongly equivalent if and only if $\alpha_1(G_1) = \alpha_2(G_1)$ and $\alpha_1(G_1^+) = \alpha_2(G_1^+)$. Together with knowing the maps described in Case 4.1, this allows one to calculate the exact number of strong equivalence classes of surjective homomorphisms from a given surface group to a given free group.

Important work of Gabai and Kazez [5], [6] which uses three-dimensional topology shows that, if $\alpha_1, \alpha_2 \colon G_1 \to G_2$ are nonzero-degree homomorphisms between infinite surface groups, they are strongly equivalent if and only if $\mathcal{G}(\alpha_1) = \mathcal{G}(\alpha_2)$, $\alpha_1(G_1) = \alpha_2(G_1)$ and $\alpha_1(G_1^+) = \alpha_2(G_1^+)$. They also show that, if $\alpha_1, \alpha_2 \colon G_1 \to G_2$ are homomorphisms between surface groups at least one of which is finite, then $\alpha_1$, $\alpha_2$ are strongly equivalent if and only if $\mathcal{G}(\alpha_1) = \mathcal{G}(\alpha_2)$, $\alpha_1(G_1) = \alpha_2(G_1)$ and $\alpha_1(G_1^+) = \alpha_2(G_1^+)$.

## 5. A WORKED EXAMPLE

In this section we will apply the algorithm to a rather trivial example to illustrate the algebraic manipulations involved.

Consider the homomorphism $\alpha \colon \langle a, b, c, d \mid (a, b)(c, d) \rangle \to \langle x, y \mid (x, y) \rangle$ induced by the homomorphism of free groups $A \colon \langle a, b, c, d \mid \ \rangle \to \langle x, y \mid \ \rangle$ determined by $(a, b, c, d) \mapsto (x, y, x, y^{-1})$.

We have

$$\begin{aligned} A((a,b)(c,d)) &= (x,y)(x,y^{-1}) = (x,y)x^{-1}yx(x,y)^{-1}x^{-1}y^{-1}x \\ &= (x,y)^{1-x^{-1}y^{-1}x}. \end{aligned}$$

Since $\alpha$ is orientation-true, Kneser's Theorem 4.8 implies that $\mathcal{G}(\alpha)$ is obtained by applying the orientation map to $1 - x^{-1}y^{-1}x$, so $\mathcal{G}(\alpha) = 0$. Thus we want to apply the algorithm to transform $A$ into a map $A'$ inducing $\alpha$, such that $A'((a,b)(c,d)) = 1$.

Form the CW-surfaces associated with the given surface group presentations, so the free group generators can be viewed as loops.

Let us subdivide $y$ into two edges, one again called $y$, and the other called $z$. We will call the vertices $u$ and $v$, so that $x$ is a loop at $v$, $y$ joins $v$ to $u$, and $z$ joins $u$ to $v$. The algorithm requires us to subdivide $x$, but, in order to keep the example simple, we shall not do this. Now we subdivide $b$ and $d$ into two edges labelled $y1$, $z1$, and $y2$, $z2$ respectively. Here the first letter indicates the image label, while, since we plan to depict the moves in planar diagrams, we also want a label to identify equal edges, and it is convenient to use integers for this identification. Similarly, we label $c$ and $d$ as $x1$ and $x2$, respectively.

We first use Construction 2.9 to get a cellular map, and hence a diagram, and then, after some simple applications of Construction 3.5 and 3.11, we can obtain the first diagram in Figure 5.1. Now we can apply the two-step