Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 43 (1997)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: 4-MANIFOLDS, GROUP INVARIANTS, AND l_2-BETTI NUMBERS
Autor: ECKMANN, Beno
DOI: https://doi.org/10.5169/seals-63279

L'Enseignement Mathématique, t. 43 (1997), p. 271–279

4-MANIFOLDS, GROUP INVARIANTS, AND l_2-BETTI NUMBERS

by Beno ECKMANN

It has been known for some time that closed 4-manifolds provide, via the fundamental group and the Euler characteristic, interesting invariants for finitely presented groups. In this short survey we describe these and more refined invariants (using also the signature of the manifold), and explain some of their significance. The invariants are not easily calculated in general, but quite good information is obtained using l_2-Betti numbers.

The topic has been developed by several authors, more or less independently. We mention Hausmann-Weinberger [H-W], Kotschick [K], Lück [L], and myself [E1], [E2]. The paper [K] contains a wealth of information on the invariants and further important references; the application of l_2-Betti numbers appears in [E2] and in [L].

1. A BASIC CONSTRUCTION

1.1. We will always denote by M a connected orientable closed 4-manifold (compact without boundary) admitting a cell decomposition. The fundamental group $G = \pi_1(M)$ is finitely presented. Indeed, homotopy classes of loops can be represented by edge-polygons and null-homotopies of these by using 2-cells. Conversely, any finitely presented group G is the fundamental group of a closed 4-manifold. If

$$G = \langle g_1, \dots, g_m \mid r_1, \dots, r_n \rangle$$

is a presentation of G, there is a standard procedure for constructing such a manifold: One first puts $M' = S^1 \times S^3 + \dots + S^1 \times S^3$, connected sum, one copy for each generator g_i of G. Then $\pi_1(M')$ is a free group on generators $g_1, \dots, g_m$. A relator, say r_1, is a word in the g_i and can be represented by a loop S^1 in M'.

A tubular neighbourhood $S^1 \times B^3$ of S^1, where B^k is the k-dimensional ball, has boundary $S^1 \times S^2$. Replacing the interior by $B^2 \times S^2$ with the same boundary yields a new 4-manifold where the element corresponding to r_1 has been killed; and similarly for the other r_i. Let M_0 be the 4-manifold thus obtained, fulfilling $\pi_1(M_0) = G$. The idea of that construction can already be found in the old book [S-T]. Much later the procedure, in a more general context, has been called "elementary surgery".

1.2. We recall that the (good old) Euler characteristic $\chi(X)$ of a finite cell complex X is the alternating sum

$$\chi(X) = \sum (-1)^i \alpha_i ,$$

where α_i is the number of i-cells. It is easily computed for M_0 above: For M' it is $2 - 2m$ since it is $= 0$ for $S^1 \times S^3$ and since it decreases by 2 in a connected sum. Under the surgery process above it increases by 2 [use the fact that for the union of two complexes X and Y with intersection Z the characteristic is $\chi(X) + \chi(Y) - \chi(Z)$; and that $\chi(B^2 \times S^2) = 2$]. Whence

$$\chi(M_0) = 2 - 2m + 2n = 2 - 2(m - n) .$$

The difference $m - n$ is called the deficiency of the presentation of G.

1.3. On the other hand the characteristic can be expressed by the Betti numbers of the cell complex X as $\sum (-1)^i \beta_i(X)$ where $\beta_i(X) = \dim_{\mathbf{R}} H_i(X; \mathbf{R})$ (and is therefore a topological invariant). Moreover the β_i of a manifold fulfill Poincaré duality, i.e. they are equal in complementary dimensions. Thus $\chi(M) = 2 - 2\beta_1(M) + \beta_2(M)$. We recall that homology in dimension 1 depends on the fundamental group G only; β_1 is the $\mathbf{Q}$-rank of G Abelianised and we write $\beta_1(G)$ for $\beta_1(M)$. Comparing with $\chi(M_0)$ above we see that the deficiency of the presentation is $\leq \beta_1(G)$. Thus there is a maximum for the deficiency of all presentations of G, called the deficiency $\operatorname{def}(G)$ of G. [For this simple side result there are, of course, much easier arguments.]

2. The Hausmann-Weinberger invariant

2.1. As seen above, the Euler characteristic of a 4-manifold M with given finitely presented fundamental group G is bounded below by $2 - 2\beta_1(G)$. The minimum of $\chi(M)$ for all such M has been considered by Hausmann-Weinberger [H-W] and denoted by $q(G)$. Using M_0 above we have the inequalities

$$2 - 2\beta_1(G) \leq q(G) \leq 2 - 2\operatorname{def}(G) .$$

2.2. EXAMPLES.

1) In [H-W] it is shown, by a simple argument, that $q(\mathbf{Z}^n) \geq 0$ for all $n \geq 1$. We return to that case later on. Here we just recall that $q(\mathbf{Z}) = q(\mathbf{Z}^2) = q(\mathbf{Z}^4) = 0$, as is easily seen by taking an appropriate M with $\chi(M) = 0$. However for $\mathbf{Z}^3$ one only gets $0 \leq q \leq 2$, the deficiency being 0.

2) For the surface group Σ_g, $g \geq 2$, i.e. the fundamental group of the closed orientable surface of genus g, one has $\mathrm{def}(\Sigma_g) = 2g-1$ and $\beta_1 = 2g$. Thus

$$2 - 4g \leq q(\Sigma_g) \leq 4 - 4g\,.$$

3) For any knot group G (the fundamental group of the complement of a classical knot in S^3) the deficiency is 1 and $\beta_1 = 1$ whence $q(G) = 0$.

4) Let G be a 2-knot-group, i.e. the fundamental group of the complement of two-dimensional knot S^2 in S^4. As for classical knots $\beta_1(G) = 1$. Surgery along the imbedded sphere S^2 produces a 4-manifold M with fundamental group G, and with $\beta_2(M) = 0$, whence $\chi M = 0$. Thus again $q(G) = 0$.

2.3. There is a topological ingredient available in 4-manifolds which has not been used, namely the signature. This has suggested a more refined group invariant associated with 4-manifolds, see the next section.

3. THE $(\chi + \sigma)$-INVARIANT

3.1. We recall that the cohomology group $H^2(M; \mathbf{R})$ is a real quadratic space, the quadratic form being given by the cup-product evaluated on the fundamental cycle of M. It is non-degenerate, and the space splits into a positive-definite and a negative-definite subspace of dimensions β_2^+ and β_2^- respectively. The difference $\beta_2^+ - \beta_2^- = \sigma(M)$ is the signature of M. Its sign clearly depends on the orientation of M and we assume the orientation chosen in such a way that $\sigma(M) \leq 0$, i.e., $\beta_2^+ \leq \beta_2^-$. Since $\beta_2 = \beta_2^+ + \beta_2^-$ the sum $\chi(M) + \sigma(M)$ is equal to $2 - 2\beta_1(G) + 2\beta_2^+(M)$, where as always $G = \pi_1(M)$. Since that sum is bounded below by $2 - 2\beta_1(G)$ depending on G only one can define an invariant $p(G)$ to be the minimum of $\chi(M) + \sigma(M)$ for all M with fundamental group G and oriented in such a way that $\sigma(M) \leq 0$. Obviously $p(G) \leq q(G)$. An equivalent way to define $p(G)$ is to take, independently of orientations, the minimum of $\chi(M) - |\sigma(M)|$.

Putting together all above inequalities we get

$$2 - 2\beta_1(G) \leq p(G) \leq q(G) \leq 2 - 2\,\mathrm{def}(G)\,.$$

3.2. It seems difficult in general to compute the value of $p(G)$ and $q(G)$, and their group-theoretic meaning is not known. We first show how one can proceed in special cases where information on $H^2(G)$, i.e. H^2 of the Eilenberg-MacLane space $K(G,1)$ is available. We then show (Section 3.3) that it is quite interesting for applications to know that the two invariants are non-negative. (This is clearly the case if $\beta_1(G) \leq 1$, in particular if G is finite).

Any 4-manifold M with $\pi_1(M) = G$ can be imbedded in a $K(G,1)$ by adding cells of dimension $2, 3, \ldots$ in order to kill the homotopy groups in dimensions ≥ 2. This yields an injective map $H^2(G;\mathbf{R}) \longrightarrow H^2(M;\mathbf{R})$. If in $H^2(G;\mathbf{R})$ the cup-product happens to be trivial then $H^2(M;\mathbf{R})$ contains an isotropic subspace of dimension $\beta_2(G)$. In that case $\beta_2^+(M)$ must be $\geq \beta_2(G)$ so that

$$p(G) \geq 2 - 2\beta_1(G) + 2\beta_2(G)\,.$$

This applies to examples in 2.2:

For the group $G = \mathbf{Z}^3$ the 3-dimensional torus is a $K(G,1)$ and the cup-product in H^2 is trivial. Since $\beta_1(G) = \beta_2(G) = 3$ we get $p(\mathbf{Z}^3) \geq 2$ whence $p(\mathbf{Z}^3) = q(\mathbf{Z}^3) = 2$.

For $G = \Sigma_g$, $g \geq 2$, the surface of genus g is a $K(G,1)$, and $\beta_1(G) = 2g$, $\beta_2(G) = 1$. Thus $p(G) \geq 4-4g$ whence $p(\Sigma_g) = q(\Sigma_g) = 4-4g$. So here the invariants are negative. Another such case is the free group F_m on $m \geq 2$ generators where one easily finds $p(F_m) = q(F_m) = 2 - 2m$.

3.3. There are several instances where the sign of the invariants yields significant information on the 4-manifolds or the groups involved. We mention three of them.

I) *Deficiency.* From the inequality in 2.1 one immediately notes that if $q(G) \geq 0$ then $\operatorname{def}(G) \leq 1$. We will return to this fact later on.

II) *Complex surfaces.* We assume that our 4-manifold M is a complex surface (complex dimension 2). Then it is known that $\chi + \sigma$ of M can be expressed in different ways: We write c_2 for the second Chern class $c_2(M)$ evaluated on M, c_1^2 for the cup-square of the first Chern class evaluated on M. Then $\chi(M) = c_2$ and $\sigma(M) = 1/3(c_1^2 - 2c_2)$ [since the signature is 1/3 of the first Pontrjagin number, which in the complex case can be expressed by the Chern classes as above]. Thus

$$\chi(M) + \sigma(M) = c_2 + 1/3(c_1^2 - 2c_2) = 1/3(c_1^2 + c_2)\,.$$

This is 4 times the holomorphic Euler characteristic $1 - g_1 + g_2$ of M by the Riemann-Roch theorem.

PROPOSITION 1. *Let M be a complex surface, and assume that its fundamental group G fulfills $p(G) \geq 0$. Then the holomorphic Euler characteristic of M is ≥ 0.*

By the Kodaira-Enriques classification it follows that M cannot be ruled over a curve of genus ≥ 2.

REMARK. The formulae above leading to the holomorphic Euler characteristic refer to the orientation of the complex surface dictated by the complex structure. Thus the argument is valid only if in *that* orientation $\sigma(M) \leq 0$. If however $\sigma(M) > 0$ then $p(G) \geq 0$ implies that $2 - 2\beta_1(G) + 2\beta_2^+{}_{\text{wrong}}(M) \geq 0$ where $\beta_2^+{}_{\text{wrong}}$ refers to the "wrong" orientation and is $= \beta_2^-(M)$. Now $\beta_2^+(M) > \beta_2^-(M)$ by assumption. Thus the result remains true; the holomorphic characteristic is > 0.

III) *Donaldson Theory.* Finitely presented groups G with $p(G) \geq 0$ and $\beta_1(G) \geq 4$ do not qualify for the Theorems A,B, and C of Donaldson [D] relating to non-simply connected topological manifolds. Indeed in these theorems the signature is assumed to be negative with $\beta_2^+ = 0$, 1 or 2. However $p(G) \geq 0$ means $2 - 2\beta_1(G) + 2\beta_2^+(M) \geq 0$, i.e. $\beta_2^+(M) \geq \beta_1(G) - 1$.

4. DEUS EX MACHINA: l_2-COHOMOLOGY

4.1. We recall in a few words the (cellular) definition of l_2-cohomology and l_2-Betti numbers, in the case of a 4-manifold M but things apply to any finite cell-complex.

Some definitions: For any countable group G let l_2G be the Hilbert space of square-integrable real functions on G, with G operating on the left, and NG the algebra of bounded G-equivariant linear operators on l_2G. A Hilbert-G-module H is a Hilbert space with isometric left G-action which admits an isometric G-equivariant imbedding into some l_2G^m (direct sum of m copies of l_2G). The projection operator ϕ of l_2G^m with image H is given by a matrix (ϕ_{kl}), $\phi_{kl} \in NG$. The "trace" $\sum \langle \phi_{kk}(1), 1 \rangle$ is the von Neumann dimension $\dim_G H$; it is a real number ≥ 0, and $= 0$ if and only if $H = 0$.

Let $\widetilde{M}$ be the universal cover of M with the cell-decomposition corresponding to that chosen in M. The square-integrable real i-cochains of $\widetilde{M}$ constitute a Hilbert space $C^i_{(2)}(\widetilde{M})$ with isometric G-action. It decomposes into the direct sum of α_i copies of l_2G, $i = 0, \ldots, 4$. As before α_i denotes the number of i-cells of M; G is the fundamental group of M acting by permutation of the cells of $\widetilde{M}$. The $C^i_{(2)}$ with the induced coboundary operators form a Hilbert-G-module chain complex. The cohomology H^i of that complex is easily identified with $H^i(M; l_2G)$, cohomology with local coefficients (see, e.g. [E2]). The *reduced* cohomology group $\overline{H}^i$ (i.e. cocycles modulo the closure of coboundaries) of that complex can be imbedded in $C^i_{(2)}$ as a G-invariant subspace and is therefore a Hilbert-G-module. Its von Neumann dimension $\dim_G \overline{H}^i(\widetilde{M})$ is the i-th l_2-Betti number $\overline{\beta}_i(M)$. It is a topological, even a homotopy, invariant of M.

4.2. Since $\dim_G C^i_{(2)} = \alpha_i$ and since the von Neumann dimension behaves like a rank, the usual Euler-Poincaré argument shows that the l_2-Betti numbers compute the Euler characteristic exactly as the ordinary Betti numbers do:

$$\chi(M) = \sum (-1)^i \overline{\beta}_i(M).$$

Moreover the $\overline{\beta}_i$ of a closed manifold fulfill Poincaré duality. Thus

$$\chi(M) = 2\overline{\beta}_0 - 2\overline{\beta}_1 + \overline{\beta}_2.$$

According to Atiyah's l_2-signature theorem [A], $\sigma(M)$ can also be expressed by appropriate l_2-Betti numbers: $\overline{H}^2(\widetilde{M})$ splits into two complementary G-invariant subspaces with von Neumann dimensions $\overline{\beta}_2^+(M)$ and $\overline{\beta}_2^-(M)$, and $\sigma(M)$ is their difference. Thus, as with ordinary Betti numbers, one has

$$\chi(M) + \sigma(M) = 2\overline{\beta}_0(G) - 2\overline{\beta}_1(G) + 2\overline{\beta}_2^+(M).$$

We now assume G to be infinite. Then $\overline{\beta}_0(G) = 0$. Indeed a 0-cocycle f in $\widetilde{M}$ is a constant and if $\widetilde{M}$ is an infinite complex f can be l_2 only if it is $= 0$.

THEOREM 2. *If for a finitely presented group G the first l_2-Betti number $\overline{\beta}_1(G)$ is 0 then the invariants $p(G)$ and $q(G)$ are non-negative.*

COROLLARY 3. *If $\overline{\beta}_1(G) = 0$ then* $\operatorname{def}(G) \leq 1$.

COROLLARY 4. *If $G = \pi_1$(complex surface M) with $\overline{\beta}_1(G) = 0$ then the holomorphic Euler characteristic of M is non-negative.*

4.3. There are many groups for which it is known that $\overline{\beta}_1(G) = 0$. A good list is given in [B-V]. We mention here three big and interesting classes of groups with that property.

1) All finitely generated amenable groups [C-G]. We recall that this class includes the virtually solvable groups, thus in particular the finitely generated Abelian groups (whence $\mathbf{Z}^n$, example 1) in 2.2). [Actually for an amenable group G with $K(G,1)$ of finite type, i.e. there is a $K(G,1)$ with finite m-skeleta, all l_2-Betti numbers are 0.]

THEOREM 5. *If G is a finitely presented amenable group then $p(G)$ and $q(G)$ are non-negative.*

2) [L1] All finitely presented groups G containing an infinite finitely generated normal subgroup N such that there is in G/N an element of infinite order. For these "Lück groups" one has the same conclusions as in the amenable case. — In [L1] the subgroup N is assumed to be finitely presented. Lück has shown later [L2] that the weaker assumption above is sufficient.

3) The statement of Theorem 5 also holds more generally for a finitely presented group G which contains a finitely generated normal subgroup N such that G/N is infinite and amenable [E2]. The proof is somewhat different: It makes use not of the universal cover but of the cover belonging to N. The amenable group G/N operates on that cover and one can use the l_2-Betti numbers relative to G/N. — A simple example is given by a group with finitely generated commutator subgroup and infinite Abelianisation.

4.4. REMARKS.

1) We note that for finitely presented infinite amenable groups, and also for groups as in 4.3, 3) above, the deficiency is ≤ 1. This can also be proved without 4-manifolds: It suffices to consider a $K(G,1)$ with 2-skeleton corresponding to a presentation of G.

2) It is well-known that a group with deficiency ≥ 2 cannot be amenable since it contains free subgroups of rank ≥ 2; see [B-P], where a stronger result is proved.

3) There is a class of groups for which $\overline{\beta}_1$ is positive: The groups G with infinitely many ends (i.e. with $H^1(G;\mathbf{Z}G)$ of infinite rank; here one takes ordinary cohomology with local coefficients). A nice proof for this can be found in [B-V]. Another approach is to use Stallings' structure theorem from which it follows that these groups contain free subgroups of rank ≥ 2 and thus are non-amenable. For non-amenable groups the Guichardet amenability criterion [G] tells that $\overline{H}^1(G;l_2G) = H^1(G;l_2G)$. The coefficient

map $H^1(G;\mathbf{Z}G) \longrightarrow H^1(G;l_2G)$ induced by the imbedding $\mathbf{Z}G \longrightarrow l_2G$ is easily seen to be injective. Since we have assumed $H^1(G;\mathbf{Z}G) \neq 0$ the result follows.

5. The vanishing of $q(G)$

5.1. Here we mention in a few words what happens when for a finitely presented group G the invariant $q(G)$ is 0. For the details and more comments we refer to the paper [E2]. We thus consider a 4-manifold M with $\pi_1(M) = G$ and $\chi(M) = 0$.

Since we restrict attention to groups with $\overline{\beta}_1(G) = 0$ the vanishing of $\chi(M)$ implies $\overline{\beta}_2(M) = 0$, whence $\overline{H}^2(\widetilde{M}) = 0$. As shown in [E2] by a spectral sequence argument it follows that $H^2(M;\mathbf{Z}G)$ is isomorphic to $H^2(G;\mathbf{Z}G)$, ordinary cohomology with local coefficients $\mathbf{Z}G$. By Poincaré duality $H^2(M;\mathbf{Z}G) = H_2(M;\mathbf{Z}G)$ which can be identified with $H_2(\widetilde{M};\mathbf{Z})$. Since $\widetilde{M}$ is simply connected, $H_2(\widetilde{M};\mathbf{Z})$ is isomorphic to the second homotopy group $\pi_2(\widetilde{M}) = \pi_2(M)$.

What about $H_3(\widetilde{M};\mathbf{Z})$? It can be identified with $H_3(M;\mathbf{Z}G)$ which, by Poincaré duality, is $\cong H^1(M;\mathbf{Z}G) = H^1(G;\mathbf{Z}G)$. This group, the "endpoint-group" of G, is known to be either 0 or $\mathbf{Z}$ or of infinite rank. As mentioned in 4.4, remark 3) the latter case is excluded by our assumption $\overline{\beta}_1(G) = 0$. The case $H^1(G;\mathbf{Z}G) = \mathbf{Z}$ is exceptional: it means that G is virtually infinite cyclic, and we exclude this. Then $H_3(\widetilde{M};\mathbf{Z}) = 0$.

5.2. We now add the assumption that $H^2(G;\mathbf{Z}G) = 0$. This is a property shared by many groups (e.g. duality groups). Then the homology groups $H_i(\widetilde{M};\mathbf{Z})$ are $= 0$ for $i = 1,2,3,4$ ($i = 4$ because $\widetilde{M}$ is an open manifold). Thus all homotopy groups of $\widetilde{M}$ are $= 0$, $\widetilde{M}$ is contractible, M is a $K(G,1)$, and the group G fulfills Poincaré duality.

THEOREM 6. *Let G be an infinite, finitely presented group, not virtually infinite cyclic, fulfilling $\overline{\beta}_1(G) = 0$ and $H^2(G;\mathbf{Z}G) = 0$, and let M be a manifold with fundamental group G. If the Euler characteristic $\chi(M) = 0$, then M is an Eilenberg-MacLane space for G and G is a Poincaré duality group of dimension* 4.

We recall that for knot groups and 2-knot groups $q(G) = 0$, see examples 3) and 4) in 2.2. Theorem 6 can only be applied to 2-knot groups which are not classical knot groups since the latter have cohomological dimension 2.

REFERENCES

[A] ATIYAH, M. Elliptic operators, discrete groups and von Neumann algebras. *Astérisque 32* (1976), 43–72.

[B-P] BAUMSLAG, G. and S.J. PRIDE. Groups with two more generators than relators. *J. London Math. Soc. 17* (1978), 425–426.

[B-V] BEKKA, M.E.B. and A. VALETTE. Group cohomology, harmonic functions and the first L^2-Betti number. Preprint; to appear in *Potential Analysis*.

[C-G] CHEEGER, J. and M. GROMOV. L^2-cohomology and group cohomology. *Topology 25* (1986), 189–215.

[D] DONALDSON, S.K. The orientation of Yang-Mills moduli spaces and 4-manifold topology. *J. Differential Geometry 26* (1987), 397–428.

[E1] ECKMANN, B. Amenable groups and Euler characteristic. *Comment. Math. Helvetici 67* (1992), 383–393.

[E2] —— Manifolds of even dimension with amenable fundamental group. *Comment. Math. Helvetici 69* (1995), 501–511.

[G] GUICHARDET, A. *Cohomologie des groupes topologiques et des algèbres de Lie.* Cedic–F. Nathan, 1980.

[H-W] HAUSMANN, J.-C. et S. WEINBERGER. Caractéristique d'Euler et groupes fondamentaux des variétés de dimension 4. *Comment. Math. Helvetici 60* (1985), 139–144.

[K] KOTSCHICK, D. Four-manifold invariants of finitely presentable groups; in *Topology, Geometry and Field Theory.* World Scientific.

[L1] LÜCK, W. L^2-Betti numbers of mapping tori and groups. *Topology 33* (1994), 203–214.

[L2] —— Hilbert modules and modules over finite von Neumann algebras and applications to L^2-invariants. Preprint, Mainz 1995; to appear in *Math. Annalen.*

[S-T] SEIFERT, H. und W. THRELFALL. *Lehrbuch der Topologie.* Teubner, 1934.

(Reçu le 3 mars 1997)

Beno Eckmann

Forschungsinstitut für Mathematik
E. T. H. Z.
CH-8092 Zürich
Switzerland

vide-leer-empty