THEOREM OF INGHAM IMPLYING THAT DIRICHLET'S L-FUNCTIONS HAVE NO ZEROS WITH REAL PART ONE

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A THEOREM OF INGHAM IMPLYING THAT DIRICHLET'S L-FUNCTIONS HAVE NO ZEROS WITH REAL PART ONE

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§ 1. INTRODUCTION

Using Landau's lemma on Dirichlet series with non-negative coefficients, A. E. Ingham in [1] proved the following theorem.

INGHAM'S THEOREM. *Let*

$$g(s) = g(s,\epsilon) = \prod_p \left(1 - \frac{\epsilon(p)}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{\epsilon(n)}{n^s} \qquad (\operatorname{Re} s > 1),$$

where the product is extended over all prime numbers p and ϵ is a bounded completely multiplicative arithmetic function. Suppose that g can be analytically continued into some domain containing the closed interval $\left[\frac{1}{2}, 1\right]$ of the real axis. Then

$$g(1) \neq 0.$$

We recall that an arithmetic function ϵ is said to be completely multiplicative if $\epsilon(mn) = \epsilon(m)\epsilon(n)$ for all positive integers m and n. Since a completely multiplicative arithmetic function ϵ is determined by the values $\epsilon(p)$ for primes p, it is immediate that ϵ is bounded if and only if $|\epsilon(p)| \leq 1$ for all primes p. Actually Ingham assumed that $|\epsilon(p)|$ is either 0 or 1 for any prime p, but his proof can easily be modified to require only that $|\epsilon(p)| \leq 1$. (Cf. [6]).

The most interesting application of Ingham's theorem is that obtained by taking $\epsilon(n) = \chi(n)n^{-i\alpha}$, where χ is a residue character modulo k and α is a real number which is different from zero if χ is the principal character. Then

$g(s) = L(s + i\alpha, \chi)$ and the theorem gives the assertion $L(1 + i\alpha, \chi) \neq 0$. The main interest in Ingham's theorem is its breadth of applicability. In contrast the familiar use of a trigonometric inequality does not cover the case in which $\alpha = 0$ and χ is a real non-principal character; that case is the very one to which Landau's lemma applies most easily (cf. §2 of [3]).

Ingham proved the theorem by establishing and using the identity

$$(*) \qquad \zeta(s)\, g(s, \epsilon)\, g(s, \eta)\, g(s, \epsilon\eta) = g(2s, \epsilon\eta) \sum_{n=1}^{\infty} E(n)\, H(n)\, n^{-s},$$

where ζ denotes the Riemann zeta function, η is another completely multiplicative arithmetic function, $\epsilon\eta$ is the pointwise product of ϵ and η, and

$$E(n) = \sum_{d|n} \epsilon(d), \quad H(n) = \sum_{d|n} \eta(d).$$

This is a generalization of a result of Ramanujan for the case $\epsilon(n) = n^a$, $\eta(n) = n^b$, where a and b are fixed complex numbers.

While the identity $(*)$ is of some interest in itself, we show that for the purpose of proving Ingham's theorem there is no reason to make the detour needed to prove $(*)$. Of course we still require Landau's lemma, which we state in the following form.

LANDAU'S LEMMA. *Suppose β and γ are real numbers with $\beta < \gamma$. Suppose that $c_n \geq 0$ for $n = 1, 2, 3, \ldots$ and that the series $\sum c_n n^{-s}$ converges for* $\operatorname{Re} s > \gamma$. *Put*

$$f(s) = \sum_{n=1}^{\infty} c_n n^{-s} \qquad (\operatorname{Re} s > \gamma).$$

If f can be analytically continued into some domain containing the closed interval $[\beta, \gamma]$ of the real axis, then the series $\sum c_n n^{-\beta}$ converges.

For a proof of Landau's lemma see [5], [4], or §2 of [2].

The proof of Ingham's theorem which we give here uses an argument similar to that used in [4] and [5], except that the argument in those two papers was phrased in such a way as to require analytic continuation into a domain containing the interval $[0, 1]$ of the real axis instead of the interval $\left[\frac{1}{2}, 1\right]$. The interval $\left[\frac{1}{2}, 1\right]$ could not be replaced in the hypothesis of Ingham's theorem by a shorter interval, i.e., one of the form $[\theta, 1]$, where $\theta > \frac{1}{2}$; this is shown by the example in which ϵ is the Liouville function λ and $g(s) = \zeta(2s)/\zeta(s) = \sum \lambda(n)\, n^{-s}$.

§2. PROOF OF INGHAM'S THEOREM

Suppose that $g(1) = 0$. We show that this assumption leads to a contradiction. We consider the function

$$F(s) = \zeta(s)^2\, g(s)\, g^*(s) \qquad (\operatorname{Re} s > 1),$$

where $g^*(s) = g(s, \overline{\epsilon})$ and $\overline{\epsilon}$ is the arithmetic function which is the complex conjugate of ϵ. Clearly $g^*(1) = \overline{g(1)} = 0$. By hypothesis g is regular along the stretch $\left[\frac{1}{2}, 1\right]$ of the real axis and so therefore is g^*, since $g^*(s) = \overline{g(\overline{s})}$. Hence F is regular on $\left[\frac{1}{2}, 1\right]$, since the double pole of ζ^2 at $s = 1$ is canceled by the zeros of g and g^* there.

Using the identity

$$(1-z)^{-1} = \exp\left(\sum_{k=1}^{\infty} z^k/k\right) \qquad (|z| < 1),$$

we obtain (for $\operatorname{Re} s > 1$)

$$\begin{aligned}
F(s) &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-2} \left(1 - \frac{\epsilon(p)}{p^s}\right)^{-1} \left(1 - \frac{\overline{\epsilon}(p)}{p^s}\right)^{-1} \\
&= \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{2 + \epsilon(p)^k + \overline{\epsilon}(p)^k}{k p^{ks}}\right) \\
&= \prod_p \left\{1 + \left(\sum_{k=1}^{\infty} \frac{2 + \epsilon(p)^k + \overline{\epsilon}(p)^k}{k p^{ks}}\right) + \frac{1}{2}\left(\sum_{k=1}^{\infty} \frac{2 + \epsilon(p)^k + \overline{\epsilon}(p)^k}{k p^{ks}}\right)^2 + \cdots\right\}.
\end{aligned}$$

Thus F has a Dirichlet series expansion

$$F(s) = \sum_{n=1}^{\infty} a(n)\, n^{-s} \qquad (\operatorname{Re} s > 1).$$

Furthermore, since

$$2 + \epsilon(p)^k + \overline{\epsilon}(p)^k = 2 + 2\operatorname{Re}\{\epsilon(p)^k\} \geq 0,$$

we have $a(n) \geq 0$ for all n.

At this point we deviate from the approach used in [4] and [5] by noting that $a(p^2) \geq 1$ for each prime p. For, since

$$F(s) = \prod_p \left(1 + \frac{2}{p^s} + \frac{3}{p^{2s}} + \cdots\right)\left(1 + \frac{\epsilon(p)}{p^s} + \frac{\epsilon(p)^2}{p^{2s}} + \cdots\right)\left(1 + \frac{\overline{\epsilon}(p)}{p^s} + \frac{\overline{\epsilon}(p)^2}{p^{2s}} + \cdots\right),$$

we find that

$$\begin{aligned} a(p^2) &= 3 + 2\epsilon(p) + 2\overline{\epsilon}(p) + \epsilon(p)^2 + \epsilon(p)\,\overline{\epsilon}(p) + \overline{\epsilon}(p)^2 \\ &= 2 - \epsilon(p)\,\overline{\epsilon}(p) + \{1 + \epsilon(p) + \overline{\epsilon}(p)\}^2 \\ &\geq 2 - |\epsilon(p)|^2 \geq 1\,. \end{aligned}$$

Thus

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^{1/2}} \geq \sum_p \frac{a(p^2)}{p} \geq \sum_p \frac{1}{p}\,.$$

In view of the divergence of $\sum p^{-1}$, it follows that $\sum a(n)\,n^{-1/2}$ diverges.

On the other hand, applying Landau's lemma with $c_n = a(n)$, $\beta = \frac{1}{2}$, $\gamma = 1$, we find that $\sum a(n)\,n^{-1/2}$ converges. This contradiction shows that the assumption $g(1) = 0$ is untenable and so the proof is complete.

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