

# §1. Introduction

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A THEOREM OF INGHAM  
IMPLYING THAT DIRICHLET'S  $L$ -FUNCTIONS  
HAVE NO ZEROS WITH REAL PART ONE

by Paul T. BATEMAN

§ 1. INTRODUCTION

Using Landau's lemma on Dirichlet series with non-negative coefficients, A. E. Ingham in [1] proved the following theorem.

INGHAM'S THEOREM. *Let*

$$g(s) = g(s, \epsilon) = \prod_p \left( 1 - \frac{\epsilon(p)}{p^s} \right)^{-1} = \sum_{n=1}^{\infty} \frac{\epsilon(n)}{n^s} \quad (\operatorname{Re} s > 1),$$

where the product is extended over all prime numbers  $p$  and  $\epsilon$  is a bounded completely multiplicative arithmetic function. Suppose that  $g$  can be analytically continued into some domain containing the closed interval  $[\frac{1}{2}, 1]$  of the real axis. Then

$$g(1) \neq 0.$$

We recall that an arithmetic function  $\epsilon$  is said to be completely multiplicative if  $\epsilon(mn) = \epsilon(m)\epsilon(n)$  for all positive integers  $m$  and  $n$ . Since a completely multiplicative arithmetic function  $\epsilon$  is determined by the values  $\epsilon(p)$  for primes  $p$ , it is immediate that  $\epsilon$  is bounded if and only if  $|\epsilon(p)| \leq 1$  for all primes  $p$ . Actually Ingham assumed that  $|\epsilon(p)|$  is either 0 or 1 for any prime  $p$ , but his proof can easily be modified to require only that  $|\epsilon(p)| \leq 1$ . (Cf. [6]).

The most interesting application of Ingham's theorem is that obtained by taking  $\epsilon(n) = \chi(n)n^{-i\alpha}$ , where  $\chi$  is a residue character modulo  $k$  and  $\alpha$  is a real number which is different from zero if  $\chi$  is the principal character. Then

$g(s) = L(s + i\alpha, \chi)$  and the theorem gives the assertion  $L(1 + i\alpha, \chi) \neq 0$ . The main interest in Ingham's theorem is its breadth of applicability. In contrast the familiar use of a trigonometric inequality does not cover the case in which  $\alpha = 0$  and  $\chi$  is a real non-principal character; that case is the very one to which Landau's lemma applies most easily (cf. §2 of [3]).

Ingham proved the theorem by establishing and using the identity

$$(*) \quad \zeta(s) g(s, \epsilon) g(s, \eta) g(s, \epsilon\eta) = g(2s, \epsilon\eta) \sum_{n=1}^{\infty} E(n) H(n) n^{-s},$$

where  $\zeta$  denotes the Riemann zeta function,  $\eta$  is another completely multiplicative arithmetic function,  $\epsilon\eta$  is the pointwise product of  $\epsilon$  and  $\eta$ , and

$$E(n) = \sum_{d|n} \epsilon(d), \quad H(n) = \sum_{d|n} \eta(d).$$

This is a generalization of a result of Ramanujan for the case  $\epsilon(n) = n^a$ ,  $\eta(n) = n^b$ , where  $a$  and  $b$  are fixed complex numbers.

While the identity (\*) is of some interest in itself, we show that for the purpose of proving Ingham's theorem there is no reason to make the detour needed to prove (\*). Of course we still require Landau's lemma, which we state in the following form.

**LANDAU'S LEMMA.** *Suppose  $\beta$  and  $\gamma$  are real numbers with  $\beta < \gamma$ . Suppose that  $c_n \geq 0$  for  $n = 1, 2, 3, \dots$  and that the series  $\sum c_n n^{-s}$  converges for  $\operatorname{Re} s > \gamma$ . Put*

$$f(s) = \sum_{n=1}^{\infty} c_n n^{-s} \quad (\operatorname{Re} s > \gamma).$$

*If  $f$  can be analytically continued into some domain containing the closed interval  $[\beta, \gamma]$  of the real axis, then the series  $\sum c_n n^{-\beta}$  converges.*

For a proof of Landau's lemma see [5], [4], or §2 of [2].

The proof of Ingham's theorem which we give here uses an argument similar to that used in [4] and [5], except that the argument in those two papers was phrased in such a way as to require analytic continuation into a domain containing the interval  $[0, 1]$  of the real axis instead of the interval  $[\frac{1}{2}, 1]$ . The interval  $[\frac{1}{2}, 1]$  could not be replaced in the hypothesis of Ingham's theorem by a shorter interval, i.e., one of the form  $[\theta, 1]$ , where  $\theta > \frac{1}{2}$ ; this is shown by the example in which  $\epsilon$  is the Liouville function  $\lambda$  and  $g(s) = \zeta(2s)/\zeta(s) = \sum \lambda(n) n^{-s}$ .