2. Two-generated one-relator groups

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2. Two-generated one-relator groups

Let G be a one-relator group with $m \geq 3$ generators, or with torsion. It is known that G has a subgroup of finite index G_0 which surjects homomorphically onto the free group $\mathbf{F}_2$ of rank 2 (see [BP 1,2]). As $\lambda_*(\mathbf{F}_2) = 3$ one has $\lambda_*(G_0) \geq 3$, and it follows from Prop. 3.3 in [SW] that $\lambda_*(G) > 1$.

In the sequel of this section we study the growth of two-generated one-relator groups.

As we remarked in the proof of Theorem 1.2 a group

$$G = \langle a, b : R(a,b) = 1 \rangle ,$$

with the relator R having zero a-exponent sum, is an HNN-extension $H^* = (H; A, B, \phi)$, where $H = \langle b_m, \ldots, b_M; R'(b_m, \ldots, b_M) = 1 \rangle$ is another one-relator group and the associated subgroups A and B are the free subgroups of H freely generated by $\{b_m, \ldots, b_{M-1}\}$ and, respectively, $\{b_{m+1}, \ldots, b_M\}$. We can distinguish 3 cases:

I) A and B are proper subgroups of H : $A \neq H \neq B$;

II) only one associated subgroup is proper: $A = H \neq B$ or $A \neq H = B$;

III) the associated subgroups coincide with the base group: $A = H = B$.

Accordingly we say that the group G is of type (I), (II) or (III).

2.1. Lemma. *Let $G = \langle a, b : R(a,b) = 1 \rangle$ where R has zero exponent sum on a. Then*

(i) G is of type (I) if and only if each of the symbols b_m and b_M occurs in R' at least two times.

(ii) G is of type (II) if and only if either b_m or b_M occurs in R' exactly once i.e., up to inversion and cyclic permutation of R'

$$R' = \begin{cases} b_M U, & \text{or} \\ b_m V, & \end{cases}$$

where $U = U(b_m, b_{m+1}, \ldots, b_{M-1})$ (respectively $V = V(b_{m+1}, \ldots, b_M)$) involves b_m (respectively b_M).

(iii) G is of type (III) if and only if both b_m and b_M occur in R' exactly once, i.e., up to a cyclic permutation and inversion of R',

$$R = W_1 b_m W_2 b_M^{\pm 1}$$

where $W_i = W_i(b_{m+1}, \ldots, b_{M-1})$, $i = 1, 2$.

Proof. (ii) Recall that an element x of a free group F is *primitive* if it can be included in a basis for F. If the basis is $\{x, y, z, \ldots\}$, then x is called a *primitive element associated with* $\{y, z, \ldots\}$.

Now $A = H$ (respectively $B = H$) if and only if R' is a primitive element associated with $\{b_m, b_{m+1}, \ldots, b_{M-1}\}$ (respectively $\{b_{m+1}, \ldots, b_M\}$) in the free group $F(b_m, b_{m+1}, \ldots, b_M)$, and this holds if and only if $R' = U_1 b_M^{\pm 1} U_2$ (respectively $R' = V_1 b_m^{\pm 1} V_2$) where $U_i = U_i(b_m, b_{m+1}, \ldots, b_{M-1})$ (respectively $V_i = V_i(b_{m+1}, \ldots, b_M)$), $i = 1, 2$. Using cyclic permutations and/or inverse operations if needed we get the conclusion.

(iii) The same arguments used for (ii) can be applied.

(i) Follows immediately by exclusion from (ii) and (iii). □

In the remaining part of this section it will be shown that all groups of exponential growth of types (II) and (III) have uniformly exponential growth (Proposition 2.7). In particular we will show that all amenable one-relator groups of exponential growth have uniformly exponential growth (Proposition 2.6).

The proof of the following statement is trivial.

2.2. Lemma. *Let G be a finitely generated group, $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ two systems of generators. Let*

$$L = \max\left\{|b_i|_A, |a_j|_B : 1 \le i \le m,\ 1 \le j \le n\right\}.$$

Then

$$\lambda_A(G)^{\frac{1}{L}} \le \lambda_B(G) \le \lambda_A(G)^L$$

$$\lambda_B(G)^{\frac{1}{L}} \le \lambda_A(G) \le \lambda_B(G)^L .$$

2.3. Lemma. *Let G be a finitely generated group such that there exists a short exact sequence*

$$1 \longrightarrow F \longrightarrow G \longrightarrow \mathbf{Z} \longrightarrow 1$$

where F is a non abelian free group. Then G has uniformly exponential growth and $\lambda_*(G) \ge \sqrt[6]{3}$.

Proof. Let A be a finite set of generators of G. Set

$$C = \left\{c \in F : \text{ there exist } a_1, a_2 \in A \cup A^{-1} \text{ such that } c = [a_1, a_2]\right\},$$
$$B = \left\{b \in F : \text{ there exist } a \in A \cup A^{-1} \cup \{1\} \text{ and } c \in C \text{ s.t. } b = aca^{-1}\right\},$$

and let F_C (respectively F_B) denote the subgroup of F generated by C (resp. B). Then F_C and F_B are free (as subgroups of a free group), $F_C \leq F_B$ and F_C is non trivial. We are going to show that F_B is not abelian.

Assume that F_C is abelian. Then there exists a simple element (i.e. not a proper power) $t \in F$ generating a cyclic subgroup $T < F$ such that $F_C < T$. If $aca^{-1} \in T$ for all $a \in A \cup A^{-1}$ and $c \in C$, then $aTa^{-1} < T$ for all $a \in A$. Thus T is normal in G, and therefore in F. But this is impossible because non abelian free groups do not have normal cyclic subgroups. It follows that there exist $a \in A \cup A^{-1}$ and $c \in C$ such that aca^{-1} does not commute with c. This shows that F_B is not abelian and

$$\lambda_{A \cup B}(G) \geq \lambda_B(F_B) \geq 3\,.$$

As $\lambda_{A \cup B}(G) \leq \lambda_A^6(G)$ by Lemma 2.2, this ends the proof. □

2.4. LEMMA. *Let G be a finitely generated group and suppose we have an exact sequence*

$$1 \longrightarrow F \longrightarrow G \longrightarrow \mathbf{Z} \longrightarrow 1$$

where F is the union of an ascending chain of free groups of rank ≥ 2. Then G has uniformly exponential growth.

Proof. Suppose that $F^{(1)} \leq F^{(2)} \leq F^{(3)} \leq \cdots \leq F^{(n)} \leq F^{(n+1)} \leq \ldots$ and $F = \bigcup_{n=1}^{\infty} F^{(n)}$. Then, with the same notations as in previous lemma, $B \subset F^{(n)}$ for n sufficiently large, and the arguments as above can be applied. □

The following statement is a reformulation of [M: Thm. 2].

2.5. LEMMA. *Consider an exact sequence of groups*

$$1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 1$$

where A is abelian and B is finitely generated. Suppose there exist $a \in A$ and $b \in B$ such that the group generated by $\{b^i a b^{-i} : i \in \mathbf{Z}\}$ is not finitely generated. Then b and ba generate a free semigroup.

2.6. PROPOSITION. *The group $\Gamma_n = \langle t, s : tst^{-1} = s^n \rangle$, $|n| \geq 2$, has uniformly exponential growth.*

Proof. Consider, for $n \in \mathbf{Z}, n \neq 0$, the abelian group

$$\mathbf{Z}\left[\frac{1}{n}\right] = \left\{\frac{k}{n^s} : k, s \in \mathbf{Z}\right\}.$$

The group Γ_n is isomorphic to the semidirect product $\mathbf{Z}\left[\frac{1}{n}\right] \times_\phi \mathbf{Z}$, where $\phi : x \longmapsto nx$.

Let A be a finite system of generators for Γ_n and suppose that there exist $\alpha, \beta \in A$ such that $\alpha \in \Gamma_n \setminus \mathbf{Z}\left[\frac{1}{n}\right]$ and $\beta \in \mathbf{Z}\left[\frac{1}{n}\right]$. Then it is easy to check that the subgroup

$$\left\langle \alpha^j \beta \alpha^{-j};\ j \in \mathbf{Z} \right\rangle \leq \mathbf{Z}\left[\frac{1}{n}\right]$$

is not finitely generated. Thus, according to the previous lemma, the set $\{\beta, \beta\alpha\}$ generates a free semigroup and setting $A' = A \cup \{\beta\alpha\}$ one gets

$$\lambda_A(\Gamma_n) \geq \sqrt{\lambda_{A'}(\Gamma_n)} \geq \sqrt{2}\,.$$

Suppose now that the generating system $A = \{a_1, \ldots, a_m\}$ is contained in $\Gamma_n \setminus \mathbf{Z}\left[\frac{1}{n}\right]$. For all $i, j = 1, \ldots, m$ one has $[a_i, a_j] \in \mathbf{Z}\left[\frac{1}{n}\right]$ and since Γ_n is not abelian there exist i_0, j_0 such that $\alpha = [a_{i_0}, a_{j_0}] \neq 0$. Setting $A'' = A \cup \{\alpha, \alpha a_1\}$ one obtains, as before,

$$\lambda_A(\Gamma_n) \geq (\lambda_{A''}(\Gamma_n))^{\frac{1}{5}} \geq 2^{\frac{1}{5}}\,. \qquad \square$$

2.7. Proposition. *If G is a two-generated one-relator group of exponential growth of type (II) or (III) then it has uniformly exponential growth.*

Proof. Consider G as an HNN-extension $H^* = (H; A, B, \phi)$ and denote by N the kernel of the homomorphism $H^* \longrightarrow \mathbf{Z}$, $h \longmapsto \sigma_t(h)$ (here $\sigma_t(h)$ denotes the sum of exponents in h of the stable letter t). Then one has the short exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow \mathbf{Z} \longrightarrow 1\,.$$

If G is of type (II) and H is free non abelian, then $N = \bigcup_{i=1}^{\infty} t^n H t^{-n}$ is an increasing union of free groups of the same rank $m \geq 2$ and Lemma 2.4 can be applied. If $H \cong \mathbf{Z}$ then $G \cong \Gamma_n$, where $|n|$ equals the index of the proper associated subgroup in H and sign(n) = sign($\phi(1)$), and the statement follows from the previous proposition.

If G is of type (III) then $N = H$ is free of finite rank ≥ 2 and Lemma 2.3 can be applied. $\square$