3. Uniformly exponential growth and growth of graded algebras

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3. Uniformly exponential growth and growth of graded algebras

In this section we describe a method of estimating growth functions of a group in terms of its graded Lie, and associative algebras defined via dimension subgroups. We begin by recalling some concepts and notations.

As in [Gri] considerations were given with respect to a Galois field $\mathbf{GF}_p$, here we modify the arguments for a field of characteristic 0, namely $\mathbf{Q}$.

Let G be a group; denote by $\mathbf{Q}[G]$ the group algebra of G over $\mathbf{Q}$, and by $\Delta \subset \mathbf{Q}[G]$ the augmentation ideal, that is the ideal generated by the elements of the form $g-1$, with $g \in G$. Recall that the *lower central series* of G is the sequence of subgroups $\{\gamma_n(G)\}_{n=1}^{\infty}$ of G defined by $\gamma_1(G) = G$ and, for $n \geq 2$, $\gamma_n(G) = [G, \gamma_{n-1}(G)]$.

The subgroup

$$G_n = \{g \in G : g - 1 \in \Delta^n\}$$

is called the *n-th dimension subgroup* of G over $\mathbf{Q}$ and it has the following characterisation due to Jennings [J] (see also [P: IV, Thm. 1.5] or [Pm: 11, Thm. 1.10])

$$G_n = \sqrt{\gamma_n(G)} := \left\{g \in G : \exists k \in \mathbf{N}, g^k \in \gamma_n(G)\right\}.$$

For any group G one defines as usual an associative graded algebra $\mathcal{A}(G)$ and two graded Lie algebras $L(G)$ and $\mathcal{L}(G)$ by

$$\mathcal{A}(G) = \bigoplus_{n=1}^{\infty} \Delta^n/\Delta^{n+1}$$

$$L(G) = \bigoplus_{n=1}^{\infty} \left[(G_n/G_{n+1}) \otimes_{\mathbf{Z}} \mathbf{Q}\right]$$

$$\mathcal{L}(G) = \bigoplus_{n=1}^{\infty} \left[(\gamma_n(G)/\gamma_{n+1}(G)) \otimes_{\mathbf{Z}} \mathbf{Q}\right]$$

(see for instance [P], [Pm]). Quillen's Theorem [Q] states that $\mathcal{A}(G)$ is the universal enveloping algebra of $L(G)$.

Assume now that G is finitely generated and set

$$a_n(G) = \dim(\Delta^n/\Delta^{n+1})$$

$$b_n(G) = \operatorname{rank}(G_n/G_{n+1})$$

$$c_n(G) = \operatorname{rank}(\gamma_n(G)/\gamma_{n+1}(G))$$

where, by rank, we mean the torsion free rank of the corresponding abelian group. Then the following relations hold

$$\sum_{n=0}^{\infty} a_n(G)z^n = \prod_{n=1}^{\infty}(1 - z^n)^{-b_n(G)} = \prod_{n=1}^{\infty}(1 - z^n)^{-c_n(G)} .$$

The first equality follows easily from Quillen's Theorem [Pm: Thm. 4.10, Chapter 3] and the second one follows from the equality $b_n(G) = c_n(G)$ as proved in [Be].

In [Be] it is also proved that

$$\limsup_{n \longrightarrow \infty} \sqrt[n]{a_n} = \limsup_{n \longrightarrow \infty} \sqrt[n]{c_n} .$$

3.1. LEMMA. *For any finite system of generators A of a group G the following inequality holds:*

$$a_n(G) \leq \gamma_A^G(n), \quad n \geq 1 .$$

Proof. For $x, y \in G$ we have

$$xy - 1 = (x - 1) + (y - 1) + (x - 1)(y - 1)$$
$$x^{-1} - 1 = -(x - 1) - (x - 1)(x^{-1} - 1)$$

so that

$$xy - 1 \equiv (x - 1) + (y - 1) \mod \Delta^2$$
$$x^{-1} - 1 \equiv -(x - 1) \mod \Delta^2 .$$

The ideal Δ^n is spanned, over $\mathbf{Q}$, by the elements of the form

$$y_1(x_1 - 1)y_2(x_2 - 1) \cdots y_n(x_n - 1)y_{n+1} ,$$

where $x_i \in G$ and $y_j \in \mathbf{Q}[G]$, $1 \leq i \leq n$, $1 \leq j \leq n + 1$. Since

$$y = \sum_{g \in G} k_g g \equiv \sum_{g \in G} k_g \mod \Delta, \ k_g \in \mathbf{Q}$$

a basis for the quotient space Δ^n/Δ^{n+1} can be chosen among the images modulo Δ^{n+1} of the elements of the form

$$(a_{i_1} - 1)(a_{i_2} - 1) \cdots (a_{i_n} - 1) ,$$

where $a_{i_j} \in A$. But $(a_{i_1} - 1)(a_{i_2} - 1) \cdots (a_{i_n} - 1) = \sum_{g \in G} k'_g g$, where the summation extends over elements g of length at most n with respect to the system of generators A. $\square$

3.2. COROLLARY. *Let G be a finitely generated group and suppose that the ranks of $\gamma_n(G)/\gamma_{n+1}(G)$ grow exponentially. Then G has uniformly exponential growth and the estimate*

$$\lambda_*(G) \geq \limsup_{n \longrightarrow \infty} \sqrt[n]{\operatorname{rank}(\gamma_n(G)/\gamma_{n+1}(G))}$$

holds.

Recall that a group G is *parafree of para-rank* m if it is residually nilpotent and the factors of consecutive groups in its lower central series equal the corresponding ones of a free group of rank m. There are parafree groups which are not isomorphic to free groups [B 2,3].

3.3. PROPOSITION. *A finitely generated parafree group G of para-rank $m \geq 2$ has uniformly exponential growth and*

$$\lambda_*(G) \geq m\,.$$

Proof. It is known (see for instance [MKS: Thms. 5.11 (Witt's Formulae) and 5.12]) that for a free group $\mathbf{F}_m$ the rank of $(\gamma_n(\mathbf{F}_m)/\gamma_{n+1}(\mathbf{F}_m))$ equals the n-th coefficient of the Maclaurin power series of the function $U(z) = 1/(1-mz)$ and the previous corollary can be applied. $\square$

Given a parafree group G of para-rank $m \geq 2$ it would be interesting to compare $\lambda_*(G)$ with $\lambda_*(\mathbf{F}_m) = 2m - 1$.

3.4. PROBLEM. *Is it true that, for a finitely generated para-free group G of para-rank $m \geq 2$ which is not free, one has $\lambda_*(G) > 2m - 1$?*

In order to formulate the next statement we recall the following

3.5. DEFINITION. An element $R \in F$ is said to be *primitive with respect to the lower central series* if, for all $n \geq 2$, it is not an n-th power modulo $\gamma_{\omega(R)+1}(F)$ where $\omega(R)$ is the weight of R. (The latter is defined by $R \in \gamma_{\omega(R)}(F)$ but $R \notin \gamma_{\omega(R)+1}(F)$.)

3.6. THEOREM ([L 1,2]). *Let R be an element of the free group F of finite rank m which is primitive with respect to the lower central series. Denote by $k = \omega(R)$ its weight and by $\langle R\rangle$ the normal closure of R in F. Let $G = F/\langle R\rangle$ and let $\mathcal{L}(F)$ and $\mathcal{L}(G)$ be the corresponding Lie algebras. Let then r be the image of R in $\mathcal{L}_k(F)$, the k-th component of $\mathcal{L}(F)$ and denote by I the ideal of $\mathcal{L}(F)$ generated by r.*

Then I is the kernel of the canonical homomorphism of $\mathcal{L}(F)$ onto $\mathcal{L}(G)$, i.e.

$$\mathcal{L}(G) = \mathcal{L}(F)/I.$$

Moreover for all $n \geq 1$ the abelian group $\mathcal{L}_n(G)$ is a torsion free group whose rank is the n-th coefficient of the Maclaurin power series of the function

$$U(z) = \frac{1}{1 - mz + z^k}.$$

4. MORE ON UNIFORMLY EXPONENTIAL GROWTH OF ONE-RELATOR GROUPS

Any two-generated one-relator group G can be presented in the form $G = \langle a, b : a^k w(a, b) = 1\rangle$ where $k \in \mathbf{Z}$ and $w(a, b)$ belongs to the commutator subgroup $[F, F]$ of the free group $F = F(a, b)$ freely generated by a and b (this follows from Lemma 1.1). Since a and b constitute a basis in $F/\gamma_2(F)$ and $[a, b]$ generates $\gamma_2(F)/\gamma_3(F)$, one can also present G in the form

$$G = \langle a, b : a^k[a, b]^l w(a, b) = 1\rangle$$

where $k, l \in \mathbf{Z}$ and $w(a, b) \in \gamma_3(F)$.

In this section we shall see that, under suitable assumptions on k, l and $w(a, b)$, the corresponding group has uniformly exponential growth.

As an application of Labute's Theorem we get the following:

4.1. PROPOSITION. *Let $G = \langle a, b : R(a, b) = 1\rangle$ be such that R is primitive with respect to $\{\gamma_n(F)\}_{n=1}^{\infty}$ and $R \in \gamma_3(F)$. Then G has uniformly exponential growth.*

Proof. If $\omega(R) \geq 3$, Theorem 3.6 shows that the corresponding function $U(z)$ has a pole z_0 with $0 < z_0 < 1$. It follows that the coefficients $c_n(G)$ grow exponentially. By Corollary 3.2, $\lambda_*(G) > 1$. □