4. More on uniformly exponential growth of one-relator groups

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **43 (1997)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **09.08.2024**

3.6. THEOREM ([L 1,2]). *Let R be an element of the free group F of finite rank m which is primitive with respect to the lower central series. Denote by $k = \omega(R)$ its weight and by $\langle R\rangle$ the normal closure of R in F. Let $G = F/\langle R\rangle$ and let $\mathcal{L}(F)$ and $\mathcal{L}(G)$ be the corresponding Lie algebras. Let then r be the image of R in $\mathcal{L}_k(F)$, the k-th component of $\mathcal{L}(F)$ and denote by I the ideal of $\mathcal{L}(F)$ generated by r.*

Then I is the kernel of the canonical homomorphism of $\mathcal{L}(F)$ onto $\mathcal{L}(G)$, i.e.

$$\mathcal{L}(G) = \mathcal{L}(F)/I.$$

Moreover for all $n \geq 1$ the abelian group $\mathcal{L}_n(G)$ is a torsion free group whose rank is the n-th coefficient of the Maclaurin power series of the function

$$U(z) = \frac{1}{1 - mz + z^k}.$$

4. MORE ON UNIFORMLY EXPONENTIAL GROWTH OF ONE-RELATOR GROUPS

Any two-generated one-relator group G can be presented in the form $G = \langle a, b : a^k w(a,b) = 1\rangle$ where $k \in \mathbf{Z}$ and $w(a,b)$ belongs to the commutator subgroup $[F,F]$ of the free group $F = F(a,b)$ freely generated by a and b (this follows from Lemma 1.1). Since a and b constitute a basis in $F/\gamma_2(F)$ and $[a,b]$ generates $\gamma_2(F)/\gamma_3(F)$, one can also present G in the form

$$G = \langle a, b : a^k [a,b]^l w(a,b) = 1\rangle$$

where $k, l \in \mathbf{Z}$ and $w(a,b) \in \gamma_3(F)$.

In this section we shall see that, under suitable assumptions on k, l and $w(a,b)$, the corresponding group has uniformly exponential growth.

As an application of Labute's Theorem we get the following:

4.1. PROPOSITION. *Let $G = \langle a, b : R(a,b) = 1\rangle$ be such that R is primitive with respect to $\{\gamma_n(F)\}_{n=1}^{\infty}$ and $R \in \gamma_3(F)$. Then G has uniformly exponential growth.*

Proof. If $\omega(R) \geq 3$, Theorem 3.6 shows that the corresponding function $U(z)$ has a pole z_0 with $0 < z_0 < 1$. It follows that the coefficients $c_n(G)$ grow exponentially. By Corollary 3.2, $\lambda_*(G) > 1$. □

For Proposition 4.3 we need the following notations. Let ξ be a positive rational number such that $\xi \neq 1$ and denote by Q_ξ the smallest subgroup of the additive group of the rationals, which contains 1 and is invariant under multiplication by ξ and ξ^{-1}. In other words if $\xi = \frac{p}{q}$ with $p, q \in \mathbf{Z}$ and $\gcd(p, q) = 1$ then $Q_\xi \equiv \mathbf{Z}[\frac{1}{p}, \frac{1}{q}]$. Consider now the automorphism α of Q_ξ defined by $\alpha(x) = \xi x$, $x \in Q_\xi$. Let $\mathbf{Z}$ act on Q_ξ by powers of α. Denote by $G_\xi = Q_\xi \rtimes_\alpha \mathbf{Z}$ the corresponding semidirect product. The group G_ξ is a two-generated group with system of generators $\{\bar{a}, \bar{b}\}$, where $\bar{a} = 1 \in Q_\xi$ and the element $\bar{b}$ implements the automorphism α : $\bar{b}^{-1} x \bar{b} = \alpha(x)$, $x \in Q_\xi$.

Let now d be a natural number ≥ 2 and set $B_d = \prod_{\mathbf{Z}} \mathbf{Z}_d$. The group $\mathbf{Z}$ acts on B_d by shifts. The corresponding semidirect product $\Gamma(d)$, also denoted by $\mathbf{Z}_d \wr \mathbf{Z}$, is called the *wreath product* of $\mathbf{Z}$ and $\mathbf{Z}_d$. We shall consider $\Gamma(d)$ as generated by $\bar{a} = (\ldots, 0, 0, 1, 0, 0, \ldots)$ where 1 denotes a generator of $\mathbf{Z}_d$ (in the expression of $\bar{a}$ it appears at the 0-th coordinate place), and by $\bar{b}$, the element which implements the shift.

We have short exact sequences

$$0 \longrightarrow Q_\xi \longrightarrow G_\xi \longrightarrow \mathbf{Z} \longrightarrow 0$$
$$0 \longrightarrow B_d \longrightarrow \Gamma(d) \longrightarrow \mathbf{Z} \longrightarrow 0$$

so that G_ξ and $\Gamma(d)$ are two-step solvable. Slightly modifying the proof of Proposition 2.6 one gets

4.2. LEMMA. *The groups G_ξ and $\Gamma(d)$ have uniformly exponential growth.*

Our last class of two-generated one-relator groups of uniformly exponential growth is determined in the following statement.

4.3. PROPOSITION. *Let $G = \langle a, b; a^k[a, b]^l w(a, b) = 1 \rangle$ with $k, l \in \mathbf{Z}$ and $w(a, b) \in F^{(2)}$ where $F^{(2)} = [[F, F], [F, F]]$ denotes the second commutator subgroup of the free group $F = F(a, b)$ on a and b. Suppose that $(k, l) \notin \{\pm(2, 1), \pm(1, 1), \pm(1, 0), \pm(0, 1)\}$. Then G has uniformly exponential growth.*

Proof. Set $G_{k,l} = \langle a, b; a^k[a, b]^l w(a, b) \rangle$. Set also

$$\bar{G}_{k,l} = \langle a, b; a^k[a, b]^l w(a, b), F^{(2)} \rangle = \langle a, b; a^k[a, b]^l, F^{(2)} \rangle$$

which is a 2-step solvable quotient group of $G_{k,l}$. We shall show that $\bar{G}_{k,l}$ can be mapped homomorphically onto either G_ξ or $\Gamma(d)$ for a suitable positive rational number $\xi \neq 1$ or natural number $d \geq 2$.

Suppose first that $k \neq l, 2l$ and $lk \neq 0$. These assumptions guarantee that $\xi := \left|\frac{l-k}{l}\right| \neq 0, 1$. Then the map $a \longmapsto (\bar{a})^{sgn(\frac{l-k}{l})}, b \longmapsto \bar{b}$ from F onto G_ξ factorizes through $\overline{G}_{k,l}$. Indeed if we suppose, for instance, that $\frac{l-k}{l} > 0$, then the image of $a^k[a, b]^l$ is the number $k + l(-1+\xi) \in \mathbf{Q}_\xi$ which is zero. Thus $\overline{G}_{k,l}$ maps onto G_ξ.

Suppose now that $\gcd(k, l) = d$ or $(k, l) \in \{\pm(d, 0), \pm(0, d)\}$ for some $d \geq 2$. Then, the same arguments as before show that $\overline{G}_{k,l}$ can be mapped onto $\Gamma(d)$ via the map $a \longmapsto \bar{a}, b \longmapsto \bar{b}$.

Finally observe that $\overline{G}_{0,0}$ is the free two-generated two-step solvable group $F/F^{(2)}$ and thus maps homomorphically onto $\Gamma(d)$ for any $d \geq 2$.

The proof follows from Lemma 4.2. □

Remark that the two-generated one-relator groups that are not covered by our statements have their relator that can be reduced to one of the form bw, $[a, b]w$ or $ba^{-1}baw$, where $w = w(a, b) \in F^{(2)}$.

Let us finish the paper by the following observation.

In [GrLP] it was conjectured that if G is a group with m generators and p relations, then

$$\lambda_*(G) \geq 2(m-p) - 1\,.$$

For one-relator groups there is one case when Gromov's conjecture holds true.

4.4. Proposition. *Let $G = \langle a_1, a_2, \ldots, a_m : R(a_1, a_2, \ldots, a_m) = 1\rangle$, with $m \geq 2$, be a one-relator group such that the relator R does not belong to the commutator subgroup F' of the free group F of rank m freely generated by $a_1, a_2, \ldots, a_m$. Then $\lambda_*(G) \geq 2m-3$.*

Proof. We may assume that G is torsion-free. Indeed if $U, V \in F$ are such $U = V^k$ for some $k \in \mathbf{Z}$, then $U \in F'$ iff $V \in F'$. If the relator R is a proper power, say $R = W^k$, then G maps onto $G_1 = \langle a_1, a_2, \ldots, a_m : W(a_1, a_2, \ldots, a_m) = 1\rangle$, which is torsion-free, and $\lambda_*(G) \geq \lambda_*(G_1)$.

Under our assumptions on R, $H_1(G, \mathbf{Q}) \cong \mathbf{Z}^{m-1}$ and the second rational homology group $H_2(G, \mathbf{Q})$ vanishes.

In [S] it is proven that if $H_2(G, \mathbf{k}) = 0$, where $\mathbf{k}$ is a field, then any subset $\{x_j\} \in G$, whose image in $H_1(G, \mathbf{k})$ is linearly independent, freely generates a free group.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite system of generators for G. Then $\overline{X} = \{\bar{x}_1, \ldots, \bar{x}_n\}$, where $\bar{x}_i$ denotes the image of x_i in $H_1(G, \mathbf{Q})$, generates

$H_1(G, \mathbf{Q})$. We can find an independent subsystem $\{\bar{x}_{i_1}, \ldots \bar{x}_{i_{m-1}}\}$ in $H_1(G, \mathbf{Q})$ such that its pre-image $\{x_{i_1}, \ldots, x_{i_m}\}$ freely generates a free group. Therefore $\lambda_X(G) \geq 2(m-1) - 1 = 2m - 3$. □

It seems to us that for a one-relator group G of rank $m \geq 3$ the inequality $\lambda_*(G) \geq 2m - 3$ cannot be deduced directly from Magnus' Theorem as it is claimed in [GrLP].

ACKNOWLEDGMENTS. This paper was written during our stay at the Section de Mathématiques de l'Université de Genève. We thank Professor Pierre de la Harpe for his invitation and kind hospitality. We also thank him for his encouragement, stimulating conversations and suggestions.

We are grateful for various supports: the CNR for the first author, the RFFI grant 96-01-00713 for the second author, and the "Fonds National Suisse de la Recherche Scientifique" for both of us.

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