4. Calculating Bott-Chern Forms

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **43 (1997)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **12.07.2024**

4. Calculating Bott-Chern Forms

In this section we will consider an exact sequence

$$\overline{\mathcal{E}}:\ 0 \to \bar{S} \to \bar{E} \to \bar{Q} \to 0,$$

where the metrics on $\bar{S}$ and $\bar{Q}$ are induced from the metric on E. Let r, n be the ranks of the bundles S and E. Let $\phi \in I(n)$ be homogeneous of degree k. We will formulate a theorem for calculating the Bott-Chern form $\widetilde{\phi}(\overline{\mathcal{E}})$. This result follows from the work of Bott-Chern, Cowen, Bismut and Gillet-Soulé.

Let ϕ' be defined as in §2. For any two matrices $A, B \in M_n(\mathbf{C})$ set

$$\phi'(A;B) := \sum_{i=1}^{k} \phi'(A, A, \dots, A, B_{(i)}, A, \dots, A),$$

where the index i means that B is in the i-th position.

Choose a local orthonormal frame $s = (s_1, s_2, \dots, s_n)$ of E such that the first r elements generate S, and let $K(\bar{S})$, $K(\bar{E})$ and $K(\bar{Q})$ be the curvature matrices of $\bar{S}$, $\bar{Q}$ and $\bar{E}$ with respect to s. Let $K_S = \frac{i}{2\pi}K(\bar{S})$, $K_E = \frac{i}{2\pi}K(\bar{E})$ and $K_Q = \frac{i}{2\pi}K(\bar{Q})$. The matrix K_E has the form

$$K_E = \left(\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right)$$

where K_{11} is an $r \times r$ submatrix. Also consider the matrices

$$K_0 = \left(\begin{array}{c|c} K_S & 0 \\ \hline K_{21} & K_Q \end{array}\right) \quad \text{and} \quad J_r = \left(\begin{array}{c|c} Id_r & 0 \\ \hline 0 & 0 \end{array}\right).$$

Let u be a formal variable and $K(u) := uK_E + (1-u)K_0$. Finally, let $\phi^!(u) = \phi'\big(K(u); J_r\big)$. We then have the following

Theorem 2.

$$\widetilde{\phi}(\overline{\mathcal{E}}) = \int_0^1 \frac{\phi^!(u) - \phi^!(0)}{u}\, du. \tag{3}$$

Proof. We prove that $\widetilde{\phi}(\overline{\mathcal{E}})$ as defined above satisfies axioms (i)-(iii) of Theorem 1. The main step is the first axiom; this was essentially done in [BC] §4, when $\phi = c$ is the total Chern class. In the form (3) (again for the total Chern class), the equation was given by Cowen in [C1] and [C2], while simplifying Bott and Chern's proof. We follow both sources in sketching a proof of this more general result.

Let h and h_Q denote the metrics on E and Q respectively. Define the orthogonal projections $P_1 : \bar{E} \to \bar{S}$ and $P_2 : \bar{E} \to \bar{Q}$ and put $h_u(v,v') = uh(P_1v, P_1v') + h(P_2v, P_2v')$ for $v, v' \in E_x$ and $0 < u \leqslant 1$. Then h_u is a hermitian norm, $h_1 = h$ and $h_u \to h_Q$ as $u \to 0$. Let $K(E,h_u)$ be the curvature matrix of (E,h_u) relative to the holomorphic frame s defined above. Proposition 3.1 of [C2] proves that $\frac{i}{2\pi}K(E,h_u) = K(u)$. It follows from Proposition 3.28 of [BC] that for $0 < t \leqslant 1$,

$$\phi(E,h_t) - \phi(E,h) = dd^c \int_t^1 \frac{\phi'\big(K(u);J_r\big)}{u}\,du\,.$$

If we could set $t = 0$ we would be done; however, the integral will not be convergent in general. Note that $K(u) = K_0 + uK_1$, where $K_1 \in A^{1,1}\big(X, \mathrm{End}(E)\big)$ is independent of u. Therefore it will suffice to show that $\phi'(K_0;J_r)$ is a closed form, so that it can be deleted from the integral. For this we may assume that $\phi = p_\lambda$ is a product of power sums, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ a partition. Then

$$\begin{aligned} p'_\lambda(K_0;J_r) &= \sum_{i=1}^m \mathrm{Tr}(K_S)^{\lambda_i - 1} \prod_{j\neq i} \big(\mathrm{Tr}(K_S) + \mathrm{Tr}(K_Q)\big)^{\lambda_j} \\ &= \sum_{i=1}^m p_{\lambda_i - 1}(\bar{S}) \prod_{j \neq i} p_{\lambda_j}(\bar{S} \oplus \bar{Q}) \end{aligned}$$

is certainly a closed form.

This proves axioms (i) and (iii); axiom (ii) is easily checked as well. □

REMARK. A similar deformation to the one in [C2] was used by Deligne in [D], 5.11 for a calculation involving the Chern character form. Special cases of Theorem 2 have been used in the literature before, see for example [GS2] Prop. 5.3, [GSZ] 2.2.3 and [Ma] Theorem 3.3.1.

We deduce some simple but useful calculations:

COROLLARY 1.

(a) $\widetilde{c_1^k}(\overline{\mathcal{E}}) = 0$ *for all* $k \geqslant 1$ *and* $\widetilde{c}_m(\overline{\mathcal{E}}) = 0$ *for all* $m > \mathrm{rk} E$.

(b) $\widetilde{p_2}(\overline{\mathcal{E}}) = 2\big(\mathrm{Tr}\,K_{11} - c_1(\bar{S})\big)$ *and* $\widetilde{c_2}(\overline{\mathcal{E}}) = c_1(\bar{S}) - \mathrm{Tr}\,K_{11}$.

Proof. (a) $c_1^!(u)$ is independent of u; hence $\widetilde{c_1}(\overline{\mathcal{E}}) = 0$. The result for higher powers of c_1 follows from Proposition 1. In addition, $\widetilde{c}_m(\overline{\mathcal{E}}) = 0$ for $m > \mathrm{rk} E$ is an immediate consequence of the definition.

(b) Using the bilinear form p_2' described previously, we find $p_2^!(u) = 2\big(u \operatorname{Tr} K_{11} + (1-u)c_1(\bar{S})\big)$, so

$$\begin{aligned}\widetilde{p_2}(\overline{\mathcal{E}}) &= 2\int_0^1 \frac{u \operatorname{Tr} K_{11} + (1-u)c_1(\bar{S}) - c_1(\bar{S})}{u}\,du \\ &= 2\big(\operatorname{Tr} K_{11} - c_1(\bar{S})\big)\,.\end{aligned}$$

To calculate $\widetilde{c_2}(\overline{\mathcal{E}})$, use the identity $2c_2 = c_1^2 - p_2$. □

Corollary 1 (b) agrees with an important calculation of Deligne's in [D], 10.1, which we now describe: Using the C^∞ splitting of $\mathcal{E}$, we can write the $\bar{\partial}$ operator for E in matrix form:

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_S & \alpha \\ 0 & \bar{\partial}_Q \end{pmatrix}, \qquad \text{for some } \alpha \in A^{0,1}\big(X, \operatorname{Hom}(Q,S)\big)\,.$$

Let $\alpha^* \in A^{1,0}\big(X, \operatorname{Hom}(S,Q)\big)$ be the transpose of α, defined using complex conjugation of forms and the metrics h_S and h_Q. If ∇ is the induced connection on $\operatorname{Hom}(Q,S)$, we can write

$$K_E = \left(\begin{array}{c|c} K_S - \frac{i}{2\pi}\alpha\alpha^* & \nabla^{1,0}\alpha \\ \hline -\nabla^{0,1}\alpha^* & K_Q - \frac{i}{2\pi}\alpha^*\alpha \end{array}\right).$$

Thus Corollary 1(b) implies that

$$\widetilde{c_2}(\overline{\mathcal{E}}) = -\frac{1}{2\pi i}\operatorname{Tr}(\alpha\alpha^*) = \frac{1}{2\pi i}\operatorname{Tr}(\alpha^*\alpha)\,,$$

and we have recovered Deligne's result. In this form the calculation was used by Moriwaki and Soulé to obtain a Bogomolov-Gieseker type inequality and a Kodaira vanishing theorem on arithmetic surfaces, respectively (see [Mo] and [S]).

The calculation of $\widetilde{c_2}$ shows that in general Bott-Chern forms are not closed. In fact, calculating $\widetilde{c_k}$ for $k \geqslant 3$ leads to much more complicated formulas, involving traces of products of curvature matrices, for which a clear geometric interpretation is lacking (unlike the matrix α above, whose negative transpose $-\alpha^*$ is the second fundamental form of $\overline{\mathcal{E}}$). In the next two sections we shall see that when $\bar{E}$ is a projectively flat bundle, the Bott-Chern forms are closed and can be calculated explicitly for any $\phi \in I(n)$.