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ATKIN-LEHNER EIGENFORMS AND STRONGLY MODULAR LATTICES

by H.-G. QUEBBEMANN

SUMMARY. There are two arithmetical objects associated with any triple (k, ℓ, χ) consisting of an even positive integer k, a squarefree positive integer ℓ, and a character χ of the group of Atkin-Lehner involutions on $\Gamma_0(\ell)$ which maps the Fricke involution to $(-1)^{k/2}$. Namely, there is a space of common Atkin-Lehner eigenforms of weight k, and there is a genus of positive definite lattices in dimension $2k$. In general, Siegel's weighted mean of the theta series from the latter genus lies in the former space. For an individual lattice, however, the same holds under the condition of strong modularity introduced in this paper. Many interesting lattices known in higher-dimensional euclidean space are strongly modular, and their theta series are explained by the theory of modular forms.

INTRODUCTION

This paper is mainly concerned with lattices on euclidean n-space that are even and similar to their duals. Let the similarity norm be ℓ and $n = 2k$. Then the theta function of such a lattice is a modular form of weight k on $\Gamma_0(\ell)$ and an eigenform of the Fricke operator. (This was used in [Qu] for primes ℓ.) When the level ℓ is composite, however, one must also take care of invariance with respect to the other Atkin-Lehner involutions. This will be our subject here.

The algebraic structure of the relevant Atkin-Lehner eigenforms turns out to be most simple when the sum of the positive divisors of ℓ divides 24. As a consequence, the notion of extremal lattices introduced by Mallows, Odlyzko and Sloane for $\ell = 1$ ([CS], Ch.7) applies in a uniform way to $\ell = 1, 2, 3, 5, 6, 7, 11, 14, 15, 23$. This explains, in particular, the theta series of quite a few remarkable lattices occurring in recent work by Nebe and Plesken ([N1], [NP]). In fact, their material has been a main stimulus to the present study.

1. Standard involutions on lattices

Throughout the paper Λ denotes an even lattice on euclidean n-space and Λ^* its dual lattice on the same space. So $x \cdot x \in 2\mathbf{Z}$ holds for all $x \in \Lambda$, and $\Lambda \subset \Lambda^*$. The group $D = \Lambda^*/\Lambda$ carries the $\mathbf{Q}/\mathbf{Z}$-valued regular quadratic form $\varphi(\overline{v}) = \frac{1}{2} v \cdot v + \mathbf{Z}$, where $\overline{v} = v + \Lambda$ for $v \in \Lambda^*$. We choose an integer $\ell > 0$ such that also $\sqrt{\ell}\Lambda^*$ is even — then Λ is said to be of *level* ℓ. It follows that $\ell\Lambda^* \subset \Lambda$, and φ takes its values in $\frac{1}{\ell}\mathbf{Z}/\mathbf{Z}$. Note that $\det \Lambda = \#D$ divides $\ell^n = (\det \Lambda)(\det \sqrt{\ell}\Lambda^*)$. We are especially interested in the situation when both factors are equal, i.e. $\det \Lambda = \ell^k$ for $n = 2k$.

Let $\ell = mm'$ with coprime integers $m, m' > 0$ (notation: $m \| \ell$), and let $D(m)$ be the m-torsion subgroup of D. Clearly, $D = D(m) \oplus D(m')$, an orthogonal decomposition with respect to the discriminant form. It is a standard procedure to associate with Λ and m that lattice between Λ and Λ^* (suitably rescaled) whose image in D is $D(m)$. We summarize its main properties.

PROPOSITION 1. *Let Λ be of level $\ell = mm'$ as above. Then also*

$$\Lambda_m = \sqrt{m}\left(\Lambda^* \cap \frac{1}{m}\Lambda\right)$$

is an even lattice of level ℓ, satisfying

$$(i) \quad \sqrt{\ell}\Lambda_m^* = \Lambda_{m'} \quad \text{and}$$
$$(ii) \quad (\Lambda_m)_m = \Lambda, \qquad (\Lambda_m)_{m'} = \Lambda_\ell .$$

If $\det \Lambda = \ell^k$ *holds for* $k = \frac{n}{2}$, *then also* $\det \Lambda_m = \ell^k$.

Proof. First Λ_m is even because $\sqrt{m}\Lambda_m \subset \Lambda$ and $\sqrt{m'}\Lambda_m \subset \sqrt{\ell}\Lambda^*$. Property (i) follows from $D(m)^\perp = D(m')$ and implies that Λ_m is of level ℓ. Also (ii) easily follows from (i). Finally $\#D = \ell^k$ implies $\#D(m) = m^k$, and so $\det \Lambda_m = \ell^k m^{n-2k}$. □

Suppose that $\det \Lambda = \ell^{n/2}$. If, moreover, Λ is isometric to Λ_m for all $m \| \ell$, then Λ is called *strongly modular*. So the classes of such lattices are the common fixed points under the group of involutions defined by Proposition 1 on the set of all classes of level ℓ.

We take a quick look at dimension $n = 2$, assuming ℓ to be squarefree, therefore $\ell \equiv 3 \pmod 4$, and using composition theory (cf. [Ca], Ch.14). An isometry class of lattices of level ℓ then corresponds to a set $\{C, C^{-1}\}$ where C is an ideal class of the ring of integers in $\mathbf{Q}(\sqrt{-\ell})$. Our group of involutions is given by the ideal classes of exponent 2 acting by multiplication. It turns out that strong modularity cannot occur unless ℓ is a prime or a product of two primes which are quadratic residues of each other. In the first case there are no nontrivial involutions, while in the second case there is precisely one fixed point $\{C, C^{-1}\}$, given by the elements of order 4 in the ideal class group (whose 2-primary part in this case is cyclic of order at least 4).

Before going on, we recall some facts on Gaussian sums; for the proofs see [Sc], Ch. 5. These invariants of quadratic forms are also most natural in connection with modular forms (see next section). We put $e(z) = e^{2\pi i z}$.

PROPOSITION 2. *Let Λ, m and $D(m)$ be as in Proposition 1. Then*

$$g_m(\Lambda) = \left(\#D(m)\right)^{-\frac{1}{2}} \sum_{\overline{v} \in D(m)} e\left(\frac{1}{2} v \cdot v\right)$$

is an eighth root of unity depending only on the isometry class of the rational quadratic space $\Lambda \otimes \mathbf{Q}$ (and on m). Furthermore,

$$g_m(\Lambda) g_{m'}(\Lambda) = g_\ell(\Lambda) = e\left(\frac{n}{8}\right) \cdot$$

When m is a prime p and $k = \dim_{\mathbf{F}_p} D(p)$,

$$g_p(\Lambda) = \begin{cases} \pm i^k & \text{if } p \equiv 3 \pmod 4 \\ \pm 1 & \text{otherwise}. \end{cases}$$

The ambiguity of signs above corresponds to the two possibilities for the isometry class of a k-dimensional regular quadratic space over $\mathbf{F}_p$. When k is even, $g_p(\Lambda) = 1$ holds if and only if $D(p)$ is hyperbolic. In general, the *genus* of Λ may be defined as the isometry class of D. In particular, for squarefree ℓ it is determined by $\det \Lambda$ and all $g_p(\Lambda)$, where p (prime) divides ℓ. The rest of this section deals with the special case

$$\ell = pq,\ p \neq q \text{ primes},\ n = 2k,\ k \text{ even},\ \det \Lambda = \ell^k .$$

Here all Λ having $g_p(\Lambda) = \varepsilon$, $g_q(\Lambda) = \delta$ form one genus, denoted by $G_n(p^\varepsilon q^\delta)$, and there are two such genera subject to the conditions $\varepsilon, \delta \in \{-1, +1\}$, $\varepsilon\delta = i^k$. As a whole, each genus is invariant under the standard involutions.

EXAMPLE 1. Let $p \equiv 3 \pmod 4$ and L be any 2-dimensional even lattice of determinant p. Then it is easy to see that the orthogonal sum $L \oplus \sqrt{q}L$ is a strongly modular lattice in $G_4(p^{\varepsilon} q^{-\varepsilon})$ for the Legendre symbol $\varepsilon = \left(\frac{-q}{p}\right)$. So we have such lattices in $G_4(2^- 3^+)$, $G_4(2^+ 7^-)$, $G_4(3^+ 5^-), \ldots$. For arbitrary primes p and q we obtain, by the same construction, a strongly modular lattice in $G_8(p^+ q^+)$ from a 4-dimensional one of level p (which always exists, cf. [Qu]).

EXAMPLE 2. Here we use some information on (proper) class numbers from [Vi], p. 153. First also $G_4(2^+ 3^-)$, $G_4(2^- 5^+)$ and $G_4(2^+ 5^-)$ contain strongly modular lattices because they have class number 1. On the other hand, let Λ be the sublattice of the D_4 root lattice formed by all $x = (x_1, \ldots, x_4)$ such that $x_1 + 2x_2 + 3x_3 \equiv 0 \pmod 7$. It is easy to see that Λ belongs to $G_4(2^- 7^+)$ and that $\min \Lambda = 4$. (As usual, $\min \Lambda$ denotes the minimum "norm" $x \cdot x$ for $x \in \Lambda$, $x \neq 0$.) Since $(0, 0, 0, 2)$ sits in $2\Lambda^* \cap \Lambda$, we have $\min \Lambda_2 = 2$, and so Λ_2 is not isometric to Λ. Since $G_4(2^- 7^+)$ has class number 2, no strongly modular lattice exists in this genus. Similarly, also $G_4(3^- 5^+)$ contains no such lattice. In this case the two classes are represented by $L \oplus L$, where L is the binary lattice with Gram matrix $\binom{2\ 1}{1\ 8}$ or $\binom{4\ 1}{1\ 4}$.

EXAMPLE 3. Again, let $p \equiv 3 \pmod 4$, q odd, and $K = \mathbf{Q}(\alpha, \beta)$, where $\alpha^2 = -p$ and $\beta^2 = (-1)^{(q-1)/2} q$. Let $\mathcal{O}_K$ be the ring of integers of K; its different is generated by $\lambda = \alpha\beta$. Given a totally positive hermitian space (V, h) of dimension r over K, we also consider it as an inner product space of dimension $n = 4r$ over $\mathbf{Q}$ for

$$x \cdot y = Tr_{K/\mathbf{Q}}\big(h(x, y)\big) .$$

Then, if Λ is an $\mathcal{O}_K$-lattice on V with hermitian dual lattice Λ^h, its euclidean dual is $\Lambda^* = \lambda^{-1} \Lambda^h$. Therefore, a unimodular hermitian lattice, when considered euclidean, belongs to the r-fold sum of the 4-dimensional genus occurring in Example 1 and is strongly modular (before rescaling, Λ_p and Λ_q are given by $\alpha^{-1}\Lambda$ and $\beta^{-1}\Lambda$, respectively). Furthermore, inspection of the traces of totally-positive elements in real quadratic fields shows that we always have $\min \Lambda \geq 4$, moreover, $\min \Lambda \geq 6$ if there is no $x \in \Lambda$ satisfying $h(x, x) = 1$ (in particular, if $r \geq 2$ and (Λ, h) is indecomposable), and even $\min \Lambda \geq 8$ if this condition holds for $q \neq 5$.

EXAMPLE 4. Similarly, $K = \mathbf{Q}\left(e\left(\frac{1}{8}\right), \sqrt{\pm q}\right)$ may be used to obtain strongly modular lattices Λ in $G_{8r}(2^+q^+)$ from rank r unimodular hermitian lattices over $\mathcal{O}_K$, setting now

$$x \cdot y = \frac{1}{2}\, Tr_{K/\mathbf{Q}}\big(h(x,y)/(2-\sqrt{2})\big)\,.$$

Again, we always have $\min \Lambda \geq 4$. E.g., $\mathcal{O}_K$ itself gives the tensor product of D_4 and the binary lattice with Gram matrix $\begin{pmatrix} 2 & 1 \\ 1 & (q+1)/2 \end{pmatrix}$.

2. ATKIN-LEHNER ACTION ON THETA FUNCTIONS

The subject treated in this section is not new, but appears to be difficult to cite from the literature (in the form we need it). For convenience, I give a rather detailed account, starting from a classical formula (due to Jacobi and others). Let Λ be an even lattice. The theta function of a coset $\overline{v} = v + \Lambda$ in Λ^* (and, in particular, Θ_Λ for $v = 0$) is that function defined on the upper half-plane by

$$\Theta_{\overline{v}}(z) = \sum_{x \in \overline{v}} e\left(\frac{1}{2}(x \cdot x)\, z\right)\,.$$

Now let $n = 2k$ (k integral), and recall that $SL_2(\mathbf{R})$ acts on functions f on the upper half-plane by

$$(f \mid_k S)(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right), \quad S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let S be in $SL_2(\mathbf{Z})$, with $c > 0$. For $u, v \in \Lambda^*$ define

$$\phi_S(u,v) = \sum_x e\big((ax \cdot x + 2x \cdot v + dv \cdot v)/2c\big)$$

where x runs through a system of representatives of those elements of $\Lambda^*/c\Lambda$ which reduce to $\overline{u}$ in $D = \Lambda^*/\Lambda$. Each summand clearly depends only on the class $x + c\Lambda$, and the whole sum depends only on $\overline{u}$ and $\overline{v}$. The latter statement is trivial for $\overline{u}$, while for $\overline{v}$ it is proved (using $1 = ad - bc$) by

$$\begin{aligned} \phi_S(u,v) &= \sum_x e\big(a(x+dv)\cdot(x+dv)/2c\big)\, e\big(-b(2x\cdot v + dv \cdot v)/2\big) \\ &= \phi_S(u+dv, 0)\, e\big(-b(2u \cdot v + dv \cdot v)/2\big)\,. \end{aligned} \tag{2.1}$$

So we may write $\phi_S(u,v) = \phi_S(\overline{u}, \overline{v})$. Then the formula we need is (see [Mi], p. 189)

(2.2) $$\Theta_{\overline{u}}\big|_k S = (\det\Lambda)^{-\frac{1}{2}}(ic)^{-k}\sum_{\overline{v}\in D}\phi_S(\overline{u},\overline{v})\Theta_{\overline{v}}\,.$$

Let Λ be of level ℓ. Then it is a well-known consequence of (2.2) that Θ_Λ is a modular form for the subgroup $\Gamma_0(\ell)$ of $SL_2(\mathbf{Z})$ defined by $c\equiv 0 \pmod{\ell}$. Now let $m\|\ell$, $m'=\ell/m$. After [AL] the m-th Atkin-Lehner involution $\Gamma_0(\ell)W_m$, a coset of $\Gamma_0(\ell)$ in its normalizer in $SL_2(\mathbf{R})$, is given by any matrix of the form

$$W_m = S\begin{pmatrix}\sqrt{m} & 0\\ 0 & 1/\sqrt{m}\end{pmatrix},\quad S=\begin{pmatrix}a\,b\\ c\,d\end{pmatrix}\in\Gamma_0(m'),\quad d\equiv 0 \,(\mathrm{mod}\; m)\,.$$

We specifically choose $a=1, c=m'$, solve $tm+t'm'=1$, and put $b=-t', d=tm$. In any case,

$$W_m^2\equiv 1,\quad W_mW_{m'}\equiv W_\ell \,\big(\mathrm{mod}\;\Gamma_0(\ell)\big)\,.$$

Now these involutions are connected, via Gaussian sums, to those in Section 1 by the following important relation which seems to have been proved first by Kitaoka ([Ki] where, however, the constant factor is not worked out; cf. also [BS], p. 77, for the statement of a generalization).

ATKIN-LEHNER IDENTITY:

(2.3) $$\Theta_\Lambda\big|_k W_m = \left(\frac{\det\Lambda_m}{\det\Lambda}\right)^{\frac{1}{4}} i^{-k} g_{m'}(\Lambda_{m'})\Theta_{\Lambda_m}\,.$$

How this identity will be used here is readily explained. Suppose that Λ (and then also Λ_m) has determinant ℓ^k. We will be interested in the following gradually stronger conditions

(1) $\Theta_\Lambda=\Theta_{\Lambda_m}$ *for all* $m\|\ell$

(2) Λ *is strongly modular.*

So condition (1) says that Θ_Λ is an eigenform of all the Atkin-Lehner involutions; Section 3 deals with such modular forms.

Proof of (2.3). Recall that here S is chosen such that $a=1$, $c=m'$ and $m|d$. Write $\overline{v}\in D$ in the form $\overline{v}=\overline{w}+\overline{y}$ where $\overline{w}\in D(m)$, $\overline{y}\in D(m')$. Then $\phi_S(0,\overline{v})=\phi_S(0,\overline{y})$ by (2.1), and (2.2) gives

$$(\det\Lambda)^{\frac{1}{2}}(im')^k\Theta_\Lambda\big|_k S = \sum_{\overline{v}\in D}\phi_S(0,\overline{v})\Theta_{\overline{v}}$$
$$= \sum_{\overline{y}\in D(m')}\phi_S(0,\overline{y})\sum_{\overline{w}\in D(m)}\Theta_{\overline{w}+\overline{y}}\,.$$

We first must know that $\phi_S(0,\bar{y}) = 0$ for $\bar{y} \neq 0$. To see this, write $x \in \Lambda$ in the form $x = x_1 + m'x'$ with x_1 from some system of representatives of $\Lambda/\sqrt{m'}\Lambda_{m'}$ and $\sqrt{m'}x' \in \Lambda_{m'}$. Then

$$\phi_S(0,y)\, e(-dy\cdot y/2m') = \sum_{x_1} e\big((x_1\cdot x_1 + 2x_1\cdot y)/2m'\big) \sum_{\overline{x'}\in D(m')} e(x'\cdot y).$$

But the sum over $D(m')$ vanishes for $\bar{y} \neq 0$ (since $D(m')$ is regular with respect to the discriminant form). It remains to determine $\phi_S(0,0)$. Put $E = \Lambda_{m'}^*/\Lambda_{m'}$. Since $(1/\sqrt{m'})\Lambda/\Lambda_{m'}$ is $E(m')$, the last formula for $y = 0$ gives

$$\begin{aligned}\phi_S(0,0) &= \#D(m') \sum_{\bar{v}\in E(m')} e\left(\frac{1}{2}v\cdot v\right)\\ &= \#D(m')\big(\#E(m')\big)^{\frac{1}{2}} g_{m'}(\Lambda_{m'})\end{aligned}$$

where

$$\begin{aligned}\#D(m')\big(\#E(m')\big)^{\frac{1}{2}} &= \big(\Lambda_{m'} : \sqrt{m'}\Lambda\big)\big(\sqrt{m'}\Lambda : m'\Lambda_{m'}\big)^{\frac{1}{2}}\\ &= \big(\#D(m')\big)^{\frac{1}{2}}(m')^k\\ &= \big(\#D(m)\big)^{-\frac{1}{2}}\big(\det\Lambda\big)^{\frac{1}{2}}(m')^k\\ &= m^{-\frac{k}{2}}\left(\frac{\det\Lambda_m}{\det\Lambda}\right)^{\frac{1}{4}}\big(\det\Lambda\big)^{\frac{1}{2}}(m')^k \qquad \square\end{aligned}$$

Note that, by Propositions 1 and 2, (2.3) in the case $\det\Lambda = \ell^k$ becomes

(2.4) $$\Theta_\Lambda\big|_k W_m = g_m(\Lambda_{m'})^{-1}\Theta_{\Lambda_m}.$$

3. Some use of modular forms

Let $W(\ell)$ be the elementary abelian 2-group formed by the Atkin-Lehner involutions $w_m = \Gamma_0(\ell)W_m$ for all $m\|\ell$. Let k be even, and let $\mathcal{S}_k(\ell)$ denote the space of cusp forms of weight k on $\Gamma_0(\ell)$. Then $W(\ell)$ acts on this space. For a character χ of $W(\ell)$ we let $\mathcal{S}_k(\ell)_\chi$ denote the subspace on which $W(\ell)$ acts by χ. If Λ and M are lattices of dimension $2k$ and level ℓ belonging to the same genus, then $f = \Theta_\Lambda - \Theta_M$ is known to be in $\mathcal{S}_k(\ell)$, and when both lattices are strongly modular identity (2.4) implies that f is in $\mathcal{S}_k(\ell)_\chi$ for the character $\chi(w_m) = g_m(\Lambda)$. So we are interested in such spaces now.

Fortunately, the dimension of $\mathcal{S}_k(\ell)_\chi$ is known; I am indebted to N.-P. Skoruppa for pointing out the reference. Let $s(\ell)$ denote the number of prime factors of ℓ, and for $m\|\ell$ let $t_k(\ell, m)$ denote the trace of w_m on $\mathcal{S}_k(\ell)$. We first have, by the orthogonality relations for characters,

$$\dim \mathcal{S}_k(\ell)_\chi = 2^{-s(\ell)} \sum_{m\|\ell} \chi(w_m) t_k(\ell, m) .$$

Now the formula for $t_k(\ell, m)$ can be found in [SZ], p. 133 (for reasons of space it will not be restated here in general). In particular, if $\ell > 3$ is squarefree, $(-1)^{k/2} 2t_k(\ell, \ell)$ turns out to be the Hurwitz class number $H(4\ell)$ (cf. [Co], Appendix B1 for a useful table), and the other $t_k(\ell, m)$ for $m > 1$ have about the same order. The only trace that grows with k is $t_k(\ell, 1)$, and so the ratio of the dimensions of $\mathcal{S}_k(\ell)_\chi$ and $\mathcal{S}_k(\ell)$ for large k is close to $2^{-s(\ell)}$. Here are the formulae evaluated for the simplest composite cases (valid for $k = 2$ only if χ is nontrivial):

$$\begin{aligned}
\dim \mathcal{S}_k(6)_\chi &= \frac{1}{4}\left(k - 3 + \big(\chi(w_2) + \chi(w_3) + \chi(w_6)\big)(-1)^{k/2}\right) \\
\dim \mathcal{S}_k(14)_\chi &= \frac{1}{2}\left(k - 2 + \big(\chi(w_7) + \chi(w_{14})\big)(-1)^{k/2}\right) \\
\dim \mathcal{S}_k(15)_\chi &= \frac{1}{2}\left(k - 2 + \big(\chi(w_5) + \chi(w_{15})\big)(-1)^{k/2}\right) .
\end{aligned}$$

Note that $\ell = 6, 14, 15$ are just those composite numbers for which $\sigma_1(\ell)$ (the sum of the positive divisors) divides 24. Let $\eta(z)$ denote the Dedekind eta function. Then for $\sigma_1(\ell)|24$ the product

$$\Delta_\ell(z) = \prod_{m|\ell} \eta(mz)^{24/\sigma_1(\ell)}$$

is known to be a cusp form on $\Gamma_0(\ell)$, with weight $k_\ell = 12\sigma_0(\ell)/\sigma_1(\ell)$, $\sigma_0(\ell) = 2^{s(\ell)}$, and nontrivial character if k_ℓ is odd. Fixing now $\ell = 6, 14$ or 15 ($k_\ell = 4, 2, 2$, resp.), we choose some 4-dimensional strongly modular lattice N of level ℓ (see Section 1, Examples 1 and 2) and define $\chi(w_m) = g_m(N)^{k/2}$ for $m|\ell$.

PROPOSITION 3. *For $\ell = 6, 14, 15$, using the notations above, a basis of $\mathcal{S}_k(\ell)_\chi$ is given by the functions*

$$\Theta_N^i \Delta_\ell^j \quad \textit{where } i \geq 0,\ j > 0,\ 2i + k_\ell j = k .$$

Proof. Recall from Section 1 that for $\ell = 6$ there are two possibilities for the genus of N, and so for the character χ when $k/2$ is odd. But in both cases the dimension formula gives the same value $(k-2)/4$. When

$\ell = 14$ and $\ell = 15$ we have seen that strong modularity uniquely determines the genus of N. So the result is an immediate consequence of the formulae above, our knowledge of χ from Section 1, and the fact that the expansion of Δ_ℓ^j at $q = e(z)$ begins with q^j. We know that Δ_ℓ belongs to our space for $k = k_\ell$ because $\dim \mathcal{S}_{k_\ell}(\ell) = 1$. $\square$

Now the notion of analytic extremality can be used for the levels above in the same way as in [CS], Ch. 7, or [Qu] when $s(\ell) = 0$ or 1 and $\sigma_1(\ell)|24$. Namely, $\Theta_N^{k/2} + \mathcal{S}_k(\ell)_\chi$ contains a unique function whose q-expansion has $1, 0, \ldots, 0$ as its first $1 + [k/k_\ell]$ coefficients, and a lattice Λ from the genus of $N^{k/2}$ is called *extremal* if Θ_Λ is this function. First examples for $\ell = 6$ and 15 occurred in Section 1.

EXAMPLE 5. Let $\ell = 14$. A strongly modular lattice Λ in $G_{4r}(2^{+}7^{\varepsilon})$, $\varepsilon = (-1)^r$, is extremal if $\min \Lambda = 2r + 2$ holds. We describe one for $r = 1$ and $r = 3$ (but suppress the trivial exercise to write down Θ_Λ in terms of Θ_N and Δ_{14}). The initial lattice N from Example 1 may be considered hermitian as $L \oplus \varrho L$ where $L = \mathcal{O}_K$, $K = \mathbf{Q}(\sqrt{-7})$, and $\varrho = \frac{1}{2}(1 + \sqrt{-7})$, a prime factor of 2 in $\mathcal{O}_K$. For $r = 1$ we let Λ be the lattice of all pairs (x, y) in $L \oplus L$ such that $x \equiv y (\mathrm{mod}\ \varrho L)$. For $r = 3$ we do the same after replacing $L = \mathcal{O}_K$ by the Barnes-Craig lattice $L = A_6^{(2)}$ (which is unimodular of rank 3 over $\mathcal{O}_K$ and has minimum norm 4). Obviously, Λ_7 and Λ_2 before rescaling are given by $(1/\sqrt{-7})\Lambda$ and $(1/\overline{\varrho})\overline{\Lambda}$, respectively, and $\min \Lambda = 2r + 2$ holds.

FURTHER EXAMPLES. For $\ell = 6$ and 15 there always exists at least one extremal lattice in dimension $n = 2k_\ell$, $4k_\ell$ and $6k_\ell$ (minimum norm 4, 6, 8, resp.), and for $\ell = 15$ there is even one in dimension $16 = 8k_\ell$ (minimum norm 10). The examples for $n > 8$ have appeared in [NP] and [N1]. The group-theoretical method used by G. Nebe to prove the strong modularity of these lattices is described in [N2]. She also has interesting strongly modular lattices for squarefree levels not satisfying $\sigma_1(\ell)|24$, some being extremal in the general sense that the theta function attains the maximal order to which a modular form in the appropriate Atkin-Lehner eigenspace takes on the value 1 at ∞. This property may sometimes be verified computationally; it seems difficult to determine the maximum in general.

The nonexistence of strongly modular lattices in certain genera like $G_4(2^{-}7^{+})$ or $G_4(3^{+}7^{-})$ may also be explained by the vanishing of the corresponding space $\mathcal{S}_2(\ell)_\chi$ and a noneven first coefficient occurring in the "genus theta series". We now discuss the latter object.

Let $\mathcal{M}_k(\ell)$ denote the space of all modular forms of even weight k on $\Gamma_0(\ell)$, and for a character χ of $W(\ell)$ let $\mathcal{M}_k(\ell)_\chi$ be the subspace on which $W(\ell)$ acts by χ. Assume that ℓ is squarefree. Then the cusps of $\Gamma_0(\ell)$ are permuted transitively by $W(\ell)$, and so the codimension of $\mathcal{S}_k(\ell)_\chi$ in $\mathcal{M}_k(\ell)_\chi$ can be at most 1. (In fact, it is 1 except for the case $k = 2$, $\chi = 1$; cf. [Mi], p. 61.) Assuming further that $\chi(w_\ell) = (-1)^{k/2}$, we have a unique genus of even lattices Λ of dimension $n = 2k$, level ℓ, determinant ℓ^k and invariants $g_m(\Lambda) = \chi(w_m)$, $m|\ell$. Let this genus be denoted by $G_n(\ell, \chi)$, and define

$$\Theta_{n,\ell,\chi} = \mu_{n,\ell,\chi}^{-1} \sum \left(\# Aut(\Lambda)\right)^{-1} \Theta_\Lambda$$

where $\mu_{n,\ell,\chi} = \sum \left(\# Aut(\Lambda)\right)^{-1}$ denotes the Minkowski-Siegel "mass", the summations being over all classes of lattices in $G_n(\ell, \chi)$. We easily see that $G_n(\ell, \chi)$ remains invariant under the standard involutions, and in general Λ and Λ_m for $m||\ell$ have the same automorphisms. Therefore, the Atkin-Lehner identity (2.4) implies that $\Theta_{n,\ell,\chi}$ lies in $\mathcal{M}_k(\ell)_\chi$. The following observation was made by A. Krieg for the case when ℓ is a prime ([Kr]).

PROPOSITION 4. *For even k, squarefree ℓ and a character χ of $W(\ell)$ satisfying $\chi(w_\ell) = (-1)^{k/2}$ we have*

$$\mathcal{M}_k(\ell)_\chi = \mathbf{C}\Theta_{2k,\ell,\chi} \oplus \mathcal{S}_k(\ell)_\chi \, .$$

Furthermore, Siegel's formula in this case is

$$\Theta_{2k,\ell,\chi}(z) = \Big(\sum_{m|\ell} \chi(w_m) m^{k/2}\Big)^{-1} \sum_{m|\ell} \chi(w_m) m^{k/2} E_k(mz)$$

where E_k denotes the normalized Eisenstein series of weight k for $SL_2(\mathbf{Z})$.

Proof. The first statement has been proved already by the preceding remarks. The second statement follows from the first one, as in the prime level case (cf. [Kr]). For the convenience of the reader the principal argument will be reproduced here. First one observes that, by a general property of genus theta functions ([An]), the decomposition above is an orthogonal one with respect to the Petersson scalar product. Since both sides of the formula to be proved have constant term 1, it then suffices to know that also the right-hand side lies in the orthogonal complement of $\mathcal{S}_k(\ell)_\chi$. When $k \geq 4$, the latter function is the image of $E_k(z)$ under the canonical $W(\ell)$-projection from $\mathcal{M}_k(\ell)$ onto $\mathcal{M}_k(\ell)_\chi$, and one is reduced to orthogonality between the classical Eisenstein series and cusp forms; the case $k = 2$ where $E_k(z)$ itself diverges must be considered separately (cf. [Mi], Ch.7). □

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H.-G. Quebbemann

Universität Oldenburg
FB 6 Mathematik
Postfach 2503
D-26111 Oldenburg
Germany

E-mail: quebbemann@mathematik.uni-oldenburg.de

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