3. Some use of modular forms

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We first must know that $\phi_S(0,\overline{y}) = 0$ for $\overline{y} \neq 0$. To see this, write $x \in \Lambda$ in the form $x = x_1 + m'x'$ with x_1 from some system of representatives of $\Lambda/\sqrt{m'}\Lambda_{m'}$ and $\sqrt{m'}x' \in \Lambda_{m'}$. Then

$$\phi_S(0,y)\, e(-dy\cdot y/2m') = \sum_{x_1} e\big((x_1\cdot x_1 + 2x_1\cdot y)/2m'\big) \sum_{\overline{x'}\in D(m')} e(x'\cdot y)\,.$$

But the sum over $D(m')$ vanishes for $\overline{y} \neq 0$ (since $D(m')$ is regular with respect to the discriminant form). It remains to determine $\phi_S(0,0)$. Put $E = \Lambda_{m'}^*/\Lambda_{m'}$. Since $(1/\sqrt{m'})\Lambda/\Lambda_{m'}$ is $E(m')$, the last formula for $y=0$ gives

$$\begin{aligned}\phi_S(0,0) &= \#D(m') \sum_{\overline{v}\in E(m')} e\left(\frac{1}{2}v\cdot v\right)\\ &= \#D(m')\big(\#E(m')\big)^{\frac12} g_{m'}(\Lambda_{m'})\end{aligned}$$

where

$$\begin{aligned}\#D(m')\big(\#E(m')\big)^{\frac12} &= \big(\Lambda_{m'} : \sqrt{m'}\Lambda\big)\big(\sqrt{m'}\Lambda : m'\Lambda_{m'}\big)^{\frac12}\\ &= \big(\#D(m')\big)^{\frac12}(m')^k\\ &= \big(\#D(m)\big)^{-\frac12}\big(\det\Lambda\big)^{\frac12}(m')^k\\ &= m^{-\frac{k}{2}}\left(\frac{\det\Lambda_m}{\det\Lambda}\right)^{\frac14}\big(\det\Lambda\big)^{\frac12}(m')^k \qquad \square\end{aligned}$$

Note that, by Propositions 1 and 2, (2.3) in the case $\det\Lambda = \ell^k$ becomes

$$\Theta_\Lambda\big|_k W_m = g_m(\Lambda_{m'})^{-1}\Theta_{\Lambda_m}\,. \tag{2.4}$$

3. Some use of modular forms

Let $W(\ell)$ be the elementary abelian 2-group formed by the Atkin-Lehner involutions $w_m = \Gamma_0(\ell)W_m$ for all $m\|\ell$. Let k be even, and let $\mathcal{S}_k(\ell)$ denote the space of cusp forms of weight k on $\Gamma_0(\ell)$. Then $W(\ell)$ acts on this space. For a character χ of $W(\ell)$ we let $\mathcal{S}_k(\ell)_\chi$ denote the subspace on which $W(\ell)$ acts by χ. If Λ and M are lattices of dimension $2k$ and level ℓ belonging to the same genus, then $f = \Theta_\Lambda - \Theta_M$ is known to be in $\mathcal{S}_k(\ell)$, and when both lattices are strongly modular identity (2.4) implies that f is in $\mathcal{S}_k(\ell)_\chi$ for the character $\chi(w_m) = g_m(\Lambda)$. So we are interested in such spaces now.

Fortunately, the dimension of $\mathcal{S}_k(\ell)_\chi$ is known; I am indebted to N.-P. Skoruppa for pointing out the reference. Let $s(\ell)$ denote the number of prime factors of ℓ, and for $m\|\ell$ let $t_k(\ell,m)$ denote the trace of w_m on $\mathcal{S}_k(\ell)$. We first have, by the orthogonality relations for characters,

$$\dim \mathcal{S}_k(\ell)_\chi = 2^{-s(\ell)} \sum_{m\|\ell} \chi(w_m) t_k(\ell,m)\,.$$

Now the formula for $t_k(\ell,m)$ can be found in [SZ], p. 133 (for reasons of space it will not be restated here in general). In particular, if $\ell > 3$ is squarefree, $(-1)^{k/2} 2t_k(\ell,\ell)$ turns out to be the Hurwitz class number $H(4\ell)$ (cf. [Co], Appendix B1 for a useful table), and the other $t_k(\ell,m)$ for $m>1$ have about the same order. The only trace that grows with k is $t_k(\ell,1)$, and so the ratio of the dimensions of $\mathcal{S}_k(\ell)_\chi$ and $\mathcal{S}_k(\ell)$ for large k is close to $2^{-s(\ell)}$. Here are the formulae evaluated for the simplest composite cases (valid for $k=2$ only if χ is nontrivial):

$$\begin{aligned}
\dim \mathcal{S}_k(6)_\chi &= \frac{1}{4}\Big(k-3+\big(\chi(w_2)+\chi(w_3)+\chi(w_6)\big)(-1)^{k/2}\Big)\\
\dim \mathcal{S}_k(14)_\chi &= \frac{1}{2}\Big(k-2+\big(\chi(w_7)+\chi(w_{14})\big)(-1)^{k/2}\Big)\\
\dim \mathcal{S}_k(15)_\chi &= \frac{1}{2}\Big(k-2+\big(\chi(w_5)+\chi(w_{15})\big)(-1)^{k/2}\Big)\,.
\end{aligned}$$

Note that $\ell = 6, 14, 15$ are just those composite numbers for which $\sigma_1(\ell)$ (the sum of the positive divisors) divides 24. Let $\eta(z)$ denote the Dedekind eta function. Then for $\sigma_1(\ell)|24$ the product

$$\Delta_\ell(z) = \prod_{m|\ell} \eta(mz)^{24/\sigma_1(\ell)}$$

is known to be a cusp form on $\Gamma_0(\ell)$, with weight $k_\ell = 12\sigma_0(\ell)/\sigma_1(\ell)$, $\sigma_0(\ell) = 2^{s(\ell)}$, and nontrivial character if k_ℓ is odd. Fixing now $\ell = 6, 14$ or 15 ($k_\ell = 4, 2, 2$, resp.), we choose some 4-dimensional strongly modular lattice N of level ℓ (see Section 1, Examples 1 and 2) and define $\chi(w_m) = g_m(N)^{k/2}$ for $m|\ell$.

PROPOSITION 3. *For $\ell = 6, 14, 15$, using the notations above, a basis of $\mathcal{S}_k(\ell)_\chi$ is given by the functions*

$$\Theta_N^i \Delta_\ell^j \quad \textit{where } i \geq 0,\ j>0,\ 2i + k_\ell j = k\,.$$

Proof. Recall from Section 1 that for $\ell = 6$ there are two possibilities for the genus of N, and so for the character χ when $k/2$ is odd. But in both cases the dimension formula gives the same value $(k-2)/4$. When

$\ell = 14$ and $\ell = 15$ we have seen that strong modularity uniquely determines the genus of N. So the result is an immediate consequence of the formulae above, our knowledge of χ from Section 1, and the fact that the expansion of Δ_ℓ^j at $q = e(z)$ begins with q^j. We know that Δ_ℓ belongs to our space for $k = k_\ell$ because $\dim \mathcal{S}_{k_\ell}(\ell) = 1$. $\square$

Now the notion of analytic extremality can be used for the levels above in the same way as in [CS], Ch. 7, or [Qu] when $s(\ell) = 0$ or 1 and $\sigma_1(\ell)|24$. Namely, $\Theta_N^{k/2} + \mathcal{S}_k(\ell)_\chi$ contains a unique function whose q-expansion has $1, 0, \ldots, 0$ as its first $1 + [k/k_\ell]$ coefficients, and a lattice Λ from the genus of $N^{k/2}$ is called *extremal* if Θ_Λ is this function. First examples for $\ell = 6$ and 15 occurred in Section 1.

EXAMPLE 5. Let $\ell = 14$. A strongly modular lattice Λ in $G_{4r}(2^{+}7^{\varepsilon})$, $\varepsilon = (-1)^r$, is extremal if $\min \Lambda = 2r + 2$ holds. We describe one for $r = 1$ and $r = 3$ (but suppress the trivial exercise to write down Θ_Λ in terms of Θ_N and Δ_{14}). The initial lattice N from Example 1 may be considered hermitian as $L \oplus \varrho L$ where $L = \mathcal{O}_K$, $K = \mathbf{Q}(\sqrt{-7})$, and $\varrho = \frac{1}{2}(1 + \sqrt{-7})$, a prime factor of 2 in $\mathcal{O}_K$. For $r = 1$ we let Λ be the lattice of all pairs (x, y) in $L \oplus L$ such that $x \equiv y (\mathrm{mod}\ \varrho L)$. For $r = 3$ we do the same after replacing $L = \mathcal{O}_K$ by the Barnes-Craig lattice $L = A_6^{(2)}$ (which is unimodular of rank 3 over $\mathcal{O}_K$ and has minimum norm 4). Obviously, Λ_7 and Λ_2 before rescaling are given by $(1/\sqrt{-7})\Lambda$ and $(1/\overline{\varrho})\Lambda$, respectively, and $\min \Lambda = 2r+2$ holds.

FURTHER EXAMPLES. For $\ell = 6$ and 15 there always exists at least one extremal lattice in dimension $n = 2k_\ell$, $4k_\ell$ and $6k_\ell$ (minimum norm 4, 6, 8, resp.), and for $\ell = 15$ there is even one in dimension $16 = 8k_\ell$ (minimum norm 10). The examples for $n > 8$ have appeared in [NP] and [N1]. The group-theoretical method used by G. Nebe to prove the strong modularity of these lattices is described in [N2]. She also has interesting strongly modular lattices for squarefree levels not satisfying $\sigma_1(\ell)|24$, some being extremal in the general sense that the theta function attains the maximal order to which a modular form in the appropriate Atkin-Lehner eigenspace takes on the value 1 at ∞. This property may sometimes be verified computationally; it seems difficult to determine the maximum in general.

The nonexistence of strongly modular lattices in certain genera like $G_4(2^{-}7^{+})$ or $G_4(3^{+}7^{-})$ may also be explained by the vanishing of the corresponding space $\mathcal{S}_2(\ell)_\chi$ and a noneven first coefficient occurring in the "genus theta series". We now discuss the latter object.

Let $\mathcal{M}_k(\ell)$ denote the space of all modular forms of even weight k on $\Gamma_0(\ell)$, and for a character χ of $W(\ell)$ let $\mathcal{M}_k(\ell)_\chi$ be the subspace on which $W(\ell)$ acts by χ. Assume that ℓ is squarefree. Then the cusps of $\Gamma_0(\ell)$ are permuted transitively by $W(\ell)$, and so the codimension of $\mathcal{S}_k(\ell)_\chi$ in $\mathcal{M}_k(\ell)_\chi$ can be at most 1. (In fact, it is 1 except for the case $k=2$, $\chi=1$; cf. [Mi], p. 61.) Assuming further that $\chi(w_\ell)=(-1)^{k/2}$, we have a unique genus of even lattices Λ of dimension $n=2k$, level ℓ, determinant ℓ^k and invariants $g_m(\Lambda)=\chi(w_m)$, $m|\ell$. Let this genus be denoted by $G_n(\ell,\chi)$, and define

$$\Theta_{n,\ell,\chi}=\mu_{n,\ell,\chi}^{-1}\sum\left(\#Aut(\Lambda)\right)^{-1}\Theta_\Lambda$$

where $\mu_{n,\ell,\chi}=\sum\left(\#Aut(\Lambda)\right)^{-1}$ denotes the Minkowski-Siegel "mass", the summations being over all classes of lattices in $G_n(\ell,\chi)$. We easily see that $G_n(\ell,\chi)$ remains invariant under the standard involutions, and in general Λ and Λ_m for $m\|\ell$ have the same automorphisms. Therefore, the Atkin-Lehner identity (2.4) implies that $\Theta_{n,\ell,\chi}$ lies in $\mathcal{M}_k(\ell)_\chi$. The following observation was made by A. Krieg for the case when ℓ is a prime ([Kr]).

PROPOSITION 4. *For even k, squarefree ℓ and a character χ of $W(\ell)$ satisfying $\chi(w_\ell)=(-1)^{k/2}$ we have*

$$\mathcal{M}_k(\ell)_\chi=\mathbf{C}\Theta_{2k,\ell,\chi}\oplus\mathcal{S}_k(\ell)_\chi\,.$$

Furthermore, Siegel's formula in this case is

$$\Theta_{2k,\ell,\chi}(z)=\Big(\sum_{m|\ell}\chi(w_m)m^{k/2}\Big)^{-1}\sum_{m|\ell}\chi(w_m)m^{k/2}E_k(mz)$$

where E_k denotes the normalized Eisenstein series of weight k for $SL_2(\mathbf{Z})$.

Proof. The first statement has been proved already by the preceding remarks. The second statement follows from the first one, as in the prime level case (cf. [Kr]). For the convenience of the reader the principal argument will be reproduced here. First one observes that, by a general property of genus theta functions ([An]), the decomposition above is an orthogonal one with respect to the Petersson scalar product. Since both sides of the formula to be proved have constant term 1, it then suffices to know that also the right-hand side lies in the orthogonal complement of $\mathcal{S}_k(\ell)_\chi$. When $k\geq 4$, the latter function is the image of $E_k(z)$ under the canonical $W(\ell)$-projection from $\mathcal{M}_k(\ell)$ onto $\mathcal{M}_k(\ell)_\chi$, and one is reduced to orthogonality between the classical Eisenstein series and cusp forms; the case $k=2$ where $E_k(z)$ itself diverges must be considered separately (cf. [Mi], Ch.7). □