THE LOCAL LINEARIZATION PROBLEM FOR SMOOTH SL(n) -ACTIONS

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THE LOCAL LINEARIZATION PROBLEM FOR SMOOTH $SL(n)$-ACTIONS

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ABSTRACT. This paper considers $SL(n,\mathbf{R})$-actions on Euclidean space fixing the origin. We show that all C^1-actions on $\mathbf{R}^n$ are linearizable. We give C^∞-actions of $SL(2,\mathbf{R})$ on $\mathbf{R}^3$ and of $SL(3,\mathbf{R})$ on $\mathbf{R}^8$ which are not linearizable. We classify the C^0-actions of $SL(n,\mathbf{R})$ on $\mathbf{R}^n$. Finally, the paper concludes with a study of the linearizability of $SL(n,\mathbf{Z})$-actions.

RÉSUMÉ. Dans cet article, on considère les actions de $SL(n,\mathbf{R})$ sur l'espace euclidien qui fixent l'origine. On montre que les actions C^1 sur $\mathbf{R}^n$ sont linéarisables. On donne des actions C^∞ de $SL(2,\mathbf{R})$ sur $\mathbf{R}^3$ et de $SL(3,\mathbf{R})$ sur $\mathbf{R}^8$ qui ne sont pas linéarisables. On classifie les actions C^0 de $SL(n,\mathbf{R})$ sur $\mathbf{R}^n$. L'article s'achève par une étude de la linéarisabilité des actions de $SL(n,\mathbf{Z})$.

1. INTRODUCTION

If a group G acts smoothly on a manifold M, fixing some point $x \in M$, then the differential of the action induces a linear action in the tangent space T_xM to M at x. The classical linearization problem is to determine whether the action of G on M is locally conjugate to its linear action on T_xM. In other words, is the action *linearizable* around x? In this paper we restrict ourselves largely to actions of $SL(n,\mathbf{R})$ on $\mathbf{R}^m$ fixing the origin: for brevity, we will simply say that $SL(n,\mathbf{R})$ acts on $(\mathbf{R}^m, 0)$.

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One of our results is:

THEOREM 1.1. *For all* $n > 1$ *and for all* $k = 1, \dots, \infty$, *every* C^k*-action of* $SL(n, \mathbf{R})$ *on* $(\mathbf{R}^n, 0)$ *is* C^k*-linearizable.*

This result is not entirely unexpected. Indeed, D'Ambra and Gromov remarked that for actions of all semi-simple groups: "at least in the C^{an}-case (and probably in the C^∞-case as well) the action is linearizable" [2, p. 98]. This was one of the main motivations of this present work. However, in [11], Guillemin and Sternberg gave an example of a C^∞-action of the *Lie algebra* $\mathfrak{sl}(2, \mathbf{R})$ on $\mathbf{R}^3$ which is not linearizable (but which does not integrate to an action of $SL(2, \mathbf{R})$). They remarked: "(the linearization theorem) is false in the C^∞ case unless some restrictions are placed on the algebra. What restrictions is unclear at present, but it seems that the algebra $\mathfrak{sl}(2, \mathbf{R})$ has to be singled out for special attention". Indeed, we show in Section 8 that Guillemin and Sternberg's example can be modified to give an action which integrates to a C^∞-action of the *group* $SL(2, \mathbf{R})$ on $\mathbf{R}^3$ which is not linearizable (even topologically). However, the moral of our results is that linearizability is not so much a function of the algebra or the group, but of the dimension in which it acts. To further this claim, we give an example, in Section 9 below, of a C^∞-action of $SL(3, \mathbf{R})$ on $\mathbf{R}^8$ which is also non-linearizable.

The paper is organized as follows. To put our results in context, we begin in Section 2 by recalling various classical linearization theorems. We state the linearization theorems of Bochner-Cartan, Sternberg and Kushnirenko, and we give proofs of the Bochner-Cartan theorem and Kushnirenko's theorem, since they are quite short. We recall Thurston's stability theorem, which we use repeatedly in this paper. We also give a proof of Hermann's result that smooth $SL(n, \mathbf{R})$-actions are formally linearizable.

In Section 3 we establish some preparatory results. In particular, we recall the notion of *suspension* (or *induction*). This is a procedure whereby, for a subgroup H of a group G and an action of H on a space M, one extends the action to an action of G on a bigger space $M' \supset M$ such that for each $x \in M$ the stabilizer of x under the action of G coincides with the stabilizer of x under the original action of H. We then use this suspension procedure to prove two results which we require later in the paper, concerning $SO(n)$-actions.

The study of $SL(n, \mathbf{R})$ actions of $\mathbf{R}^n$ is done in two parts. The case $n \geq 3$ is treated in Section 4. We prove part of Theorem 1.1 here, and in the continuous case, *we give an explicit recipe for constructing all* C^0*-actions on* $\mathbf{R}^n$: see Theorem 4.1.

In Section 5, we pause to recall some details of the adjoint representation of $SL(2,\mathbf{R})$ on its Lie algebra. Then in Section 6 we treat the linearizability of smooth $SL(2,\mathbf{R})$-actions on $\mathbf{R}^2$ and the classification of C^0-actions on $\mathbf{R}^2$. Actions of $SL(2,\mathbf{R})$ on $\mathbf{R}^m$ for $m > 2$ are quite prolific. We give *several examples* in Section 7. Then in Section 8 we give our *variation of Guillemin and Sternberg's example*. By using the method of suspension we show, in Section 9, that *one can construct a non-linearizable C^∞-action of $SL(3,\mathbf{R})$ on $\mathbf{R}^8$*. This is also constructed from the adjoint representation.

The paper concludes in Section 10 with a study of the linearizability of $SL(n,\mathbf{Z})$-actions (and more generally of lattices in semi-simple groups). We show in particular:

THEOREM 1.2.

(a) *For no values of n and m with $n > m$, are there any faithful C^1-actions of $SL(n,\mathbf{Z})$ on $(\mathbf{R}^m, 0)$.*

(b) *There is a C^∞-action of $SL(3,\mathbf{Z})$ on $(\mathbf{R}^8, 0)$ which is not topologically linearizable.*

(c) *There is a C^ω-action of $SL(2,\mathbf{Z})$ on $(\mathbf{R}^2, 0)$ which is not linearizable.*

(d) *For all $n > 2$ and $m > 2$, every C^ω-action of $SL(n,\mathbf{Z})$ on $(\mathbf{R}^m, 0)$ is C^ω-linearizable.*

Throughout this paper, by a "C^k-action" we mean an action by C^k-diffeomorphisms which is continuous in the C^k-topology. To fix ideas, we make the following explicit definition:

DEFINITION 1.3. Consider a C^1-action Φ of a group G on $(\mathbf{R}^m, 0)$ and simply denote by $g(x)$ the action of the element $g \in G$ on the point $x \in \mathbf{R}^m$. For $g \in G$, let $D(g) \in GL(m,\mathbf{R})$ denote the differential of the diffeomorphism $x \mapsto g(x)$ at the origin. Then Φ is *linearizable* if there are open neighbourhoods U, V of the origin, and a homeomorphism $F\colon (U,0) \to (V,0)$, such that for each $g \in G$ the maps

$$x \mapsto F\big(g\big(F^{-1}(x)\big)\big) \quad \text{and} \quad x \mapsto D(g)\,(x)$$

have the same germ at the origin. If Φ and F are C^k (resp. C^∞, resp. C^ω) then we say that the action is C^k- (resp. C^∞-, resp. C^ω-) linearizable.

REMARK 1.4. Notice that "linearizable" really means "locally linearizable". We don't consider the question of global linearizability since, even under the strongest hypotheses, global linearizability is too much to expect. For example, the action by conjugation of $PSL(2,\mathbf{R})$ on its universal cover $\widetilde{SL}(2,\mathbf{R}) \cong \mathbf{R}^3$ is analytic and locally linearizable, by the exponential map of the Lie algebra, but it is not globally linearizable because it has countably many fixed points (corresponding to the infinite discrete centre). In fact, even for algebraic actions, global linearization is not guaranteed [38]. Throughout this paper we will use the word *local* to mean "in some neighbourhood of the origin". We make the point however that in the case of a locally linearizable action, each homeomorphism of the action has its own domain on which it is linearizable, but there may be no common open domain for the entire group.

Note that we could also deal with *local group actions*; that is, maps Φ from some open neighbourhood of $(\mathrm{Id}, 0) \in G \times \mathbf{R}^m$ to some neighbourhood of $0 \in \mathbf{R}^m$ which satisfy the same conditions as for actions but only in the neighbourhood of $(\mathrm{Id}, 0) \in G \times \mathbf{R}^m$. There would be no essential changes in what follows.

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2. BACKGROUND AND MOTIVATION

The introduction to [21] begins: "The subject of smooth transformation groups has been strongly influenced by the following two problems: the smooth linearization problem (Is every smooth action of a compact Lie group on Euclidean space conjugate to a linear action ?), and the smooth fixed point problem (Does every smooth action of a compact Lie group on Euclidean space have a fixed point ?)." Indeed, for *compact* group actions, one has the following theorem of Salomon Bochner and Henri Cartan:

BOCHNER-CARTAN THEOREM (see [30, Chap. V, Theorem 1]). *For all $k = 1, \ldots, \infty$, every C^k-action of a compact group G on $(\mathbf{R}^m, 0)$ is C^k-linearizable.*

Proof. For each element $g \in G$, let $D(g)$ denote the differential of the action of g at the origin. Consider the map $F\colon \mathbf{R}^m \to \mathbf{R}^m$, defined by

$$F(x) = \int_G D(g)^{-1}\big(g(x)\big)\, d\mu$$

where μ is the normalized Haar measure on G. At the origin, the differential $D(F)$ is the identity map. So F is a local C^k-diffeomorphism in some neighbourhood of the origin. For each $h \in G$ one has

$$\begin{aligned} F\big(h(x)\big) &= \int_G D(g)^{-1}\big(gh(x)\big)\, d\mu = \int_G D(gh^{-1})^{-1}\big(g(x)\big)\, d\mu \\ &= \int_G D(h)\, D(g)^{-1}\big(g(x)\big)\, d\mu = D(h)\,\big(F(x)\big)\,, \end{aligned}$$

for all $x \in \mathbf{R}^m$. So locally, F conjugates h to its linear part $D(h)$. □

REMARK 2.1. The same idea shows the following: suppose a group G acts on $(\mathbf{R}^m, 0)$ by C^k diffeomorphisms and contains a finite index subgroup G_0 which is C^k-linearizable. Then the action of G is C^k-linearizable. Indeed, we can assume that the action of G_0 is linear and we observe that $D(g)^{-1}\big(g(x)\big)$ depends only on the class $[g]$ of g in $G_0\backslash G$. Therefore we can define $F\colon \mathbf{R}^m \to \mathbf{R}^m$ by

$$F(x) = \sum_{[g]\in G_0\backslash G} D(g)^{-1}\big(g(x)\big)\,.$$

This F linearizes the action of G.

REMARK 2.2. The above theorem does not hold for C^0-actions. Indeed, here are two examples. First, recall that Bing constructed a continuous involution of S^3 whose fixed point set is the "horned sphere" [4] (see [5] for other examples). Removing one of these fixed points, one obtains a $\mathbf{Z}/2\mathbf{Z}$-action on $\mathbf{R}^3$ which is not locally topologically conjugate to a linear action, because the fixed point set is not locally flat.

Secondly, we give a non-linearizable action of $S^1 = SO(2)$, since we will be interested in $SO(n)$-actions later in the paper. Let M be any compact manifold with the same homotopy type as $\mathbf{C}P^n$, for some $n \geq 3$. By pulling back the Hopf fibration $S^1 \to S^{2n+1} \to \mathbf{C}P^n$, one obtains an S^1-bundle $\widehat{M} \to M$. Here $\widehat{M}$ is a compact manifold with the same homotopy type as S^{2n+1}, as one can see by applying the 5-lemma to the long exact homotopy sequence of the two fibrations. Hence, by Smale's proof of the generalized Poincaré conjecture [39], $\widehat{M}$ is homeomorphic to S^{2n+1}. Taking the cone of S^{2n+1} we obtain an S^1-action on $(\mathbf{R}^{2n+2}, 0)$. Locally, in a punctured neighbourhood of the origin, the orbit space of this action is homeomorphic to $M \times \mathbf{R}$. Now M may be chosen to be not homeomorphic to $\mathbf{C}P^n$ [15, 29]. Then, by the h-cobordism theorem [18, Essay 3], M is not h-cobordant to $\mathbf{C}P^n$ and consequently $M \times \mathbf{R}$ is not homeomorphic to $\mathbf{C}P^n \times \mathbf{R}$. Hence the S^1-action is not locally topologically conjugate to a linear action. Indeed, a linear action of $SO(2)$ on $\mathbf{R}^{2n+2}$ which is *free* outside the origin is linearly conjugate to a product of $n+1$ copies of the canonical action of $SO(2)$ on $\mathbf{R}^2$ and its local orbit space is homeomorphic to $\mathbf{C}P^n \times \mathbf{R}$.

In fact, for actions of noncompact groups, linearization results date back to Poincaré's work on analytic maps [34]. Recall that an element L of $GL(m, \mathbf{R})$ is called *hyperbolic* if all its eigenvalues $\lambda_1, \ldots, \lambda_m$ have modulus different from one. One says that L has a *resonance* if there is some relation of the form $\lambda_i = \lambda_1^{k_1} \lambda_2^{k_2} \cdots \lambda_m^{k_m}$ where $1 \leq i \leq m$ and the k_j are non negative integers whose sum is bigger than 1. In the smooth case, one has the celebrated Sternberg Theorem:

THEOREM 2.3 ([43]). *In a neighbourhood of a fixed point, every C^∞-map whose linear part is hyperbolic and has no resonances is C^∞-linearizable.*

In the same vein, the Grobman-Hartman theorem says that in a neighbourhood of a hyperbolic fixed point, C^1-maps are topologically linearizable. See [17, Chap. 6] for a presentation of these results. Sternberg also proved in [43] that in a neighbourhood of a hyperbolic fixed point, every C^k-map whose linear part has no resonances is C^l-linearizable. Here l depends on the eigenvalues of the linear part and in general is less that k. According to [41], for the particular case of maps of the real line, one may take $l = k-1$. Here "hyperbolic" simply means that the derivative is a dilation (*i.e.* a linear map $x \mapsto ax$ with $|a| \neq 0, 1$). In fact, by [42, Theorem 4], for $k \geq 2$ one may take $l = k$ in this case. According to [12], even for $k = 1$, this last result is true but we could not locate a proof in the literature. All linearization results for maps pass immediately over to the case of flows, due the following lemma of Sternberg:

LEMMA 2.4 ([42, Lemma 4]). *Let $k = 1, \ldots, \infty$ and suppose that one has a C^k-flow ϕ^t on $(\mathbf{R}^m, 0)$. If ϕ^α is C^k-linearizable for some $\alpha \neq 0$, then ϕ^t is C^k-linearizable.*

Proof. Suppose that ϕ^α is linear. Then set

$$F(x) = \int_0^\alpha D(\phi^t)^{-1}\big(\phi^t(x)\big)\,dt\,,$$

and imitate the proof of the Bochner-Cartan Theorem. □

In particular, this gives the following result, which we will require later.

THEOREM 2.5. *Let $k = 1, \ldots, \infty$ and suppose that one has a C^k-flow ϕ^t on $(\mathbf{R}, 0)$ whose linear part $D(\phi^1)$ is a dilation. Then ϕ^t is C^k-linearizable.*

According to Guillemin and Sternberg [11], it was Palais and Smale who suggested extending the Bochner-Cartan theorem to noncompact Lie groups. Indeed, analytic actions of semi-simple Lie groups are also linearizable, as proved by Kushnirenko [22], and independently by Guillemin and Sternberg [11] (see also [10, 20, 24, 26]). In particular, one has:

THEOREM 2.6. *Every analytic action of $SL(n, \mathbf{R})$ on $(\mathbf{R}^m, 0)$ is analytically linearizable.*

Proof. The proof that we sketch is slightly simpler that the one given in [11, 22]. It uses the famous unitary trick but does not use Poincaré's linearization theorem. First complexify the analytic $SL(n, \mathbf{R})$-action to obtain a local holomorphic action of $SL(n, \mathbf{C})$ on a neighbourhood of the origin in $\mathbf{C}^n$. Now restrict this action to the action of $SU(n)$. From the proof of the Bochner-Cartan theorem, we have on some neighbourhood U of the origin, a holomorphic map $F\colon (U, 0) \to (\mathbf{C}^m, 0)$ such that

$$F\big(g(x)\big) = D(g)\,\big(F(x)\big), \quad \text{for all } g \in SU(n),\ x \in g^{-1}(U) \cap U\,,$$

where D is the differential of the action at the origin. Now fix $x \in \mathbf{C}^n$ and consider the set

$$S = \{g \in SL(n, \mathbf{C}) : F\big(g(x)\big) = D(g)\,\big(F(x)\big), \text{ on some neighbourhood of } 0\}\,.$$

This is a complex Lie subgroup of $SL(n, \mathbf{C})$ containing $SU(n)$. So, since $\mathfrak{sl}(n, \mathbf{C}) = \mathfrak{su}(n) \oplus i\,.\,\mathfrak{su}(n)$, one has $S = SL(n, \mathbf{C})$. Thus the action of $SL(n, \mathbf{C})$ is holomorphically linearizable.

Finally, F leaves $\mathbf{R}^m$ invariant and hence defines an analytic map which conjugates the action of $SL(n, \mathbf{R})$ to its linear part. □

Here is another important result:

THURSTON'S STABILITY THEOREM ([45]). *Let G be a connected Lie group or a finitely generated discrete group and suppose we have a non-trivial C^1-action of G on $(\mathbf{R}^m, 0)$. If G acts trivially on the tangent space $T_0\mathbf{R}^m$, then $H^1(G, \mathbf{R}) \neq 0$.*

REMARK 2.7. In the statement of Thurston's stability theorem, $H^*(G, \mathbf{R})$ denotes the continuous cohomology; so $H^1(G, \mathbf{R})$ is just the space of continuous homomorphisms from G to $\mathbf{R}$. Since $SL(n, \mathbf{R})$ is a simple Lie group, one has $H^1\big(SL(n, \mathbf{R}), \mathbf{R}\big) = 0$, for all n. For $n \geq 3$, $SL(n, \mathbf{Z})$ is a perfect group [32, Theorem VII.5], and so $H^1\big(SL(n, \mathbf{Z}), \mathbf{R}\big) = 0$. More generally, if Γ is a lattice in $SL(n, \mathbf{R})$, for some $n \geq 3$, then Γ has Kazhdan's property T and so $H^1(\Gamma, \mathbf{R}) = 0$ (see [50, Theorem 7.1.4 and Corollary 7.1.7]). If Γ is a lattice in $SL(2, \mathbf{R})$, then Γ doesn't have Kazhdan's property T (see [25, Proposition 3.1.9]) and $H^1(\Gamma, \mathbf{R})$ may be zero or non-zero, depending upon Γ. However, $H^1\big(SL(2, \mathbf{Z}), \mathbf{R}\big) = 0$, as the derived subgroup of $SL(2, \mathbf{Z})$ has finite index.

Note that the previous theorem can be regarded as a linearization result: for $G = SL(n, \mathbf{R})$, since $H^1(G, \mathbf{R}) = 0$, it says that the action is linearizable (trivially) if the differential at the origin is trivial.

The main point of this paper is to discuss to what extent the following theorem of Hermann can be generalized.

THEOREM 2.8 ([13]). *Every C^∞-action of $SL(n, \mathbf{R})$ on $(\mathbf{R}^m, 0)$ is formally linearizable.*

Before proving this theorem, let us recall some concepts and notation. Firstly, if $i = (i_1, \ldots, i_m)$, where $i_1, \ldots, i_m \geq 0$, and if $x = (x_1, \ldots, x_m) \in \mathbf{R}^m$, then we write $|i| = \sum_{j=1}^m i_j$ and we denote $\prod_{j=1}^m x_j^{i_j}$ by x^i. Now consider a formal power series

$$f(x) = \sum_i f_i x^i,$$

where $f_i \in \mathbf{R}^m$ for each i, and suppose that f has zero constant term (that is, $f(0) = 0$). The k^{th} Taylor polynomial of f is $T^k f = \sum_{|i| \leq k} f_i x^i$. We say that such a formal power series is a *formal diffeomorphism* of $(\mathbf{R}^m, 0)$ if $T^1 f$ defines a nonsingular linear map on $\mathbf{R}^m$. Let $\widehat{\mathrm{Diff}}(\mathbf{R}^m, 0)$ denote the group of formal diffeomorphisms of $\mathbf{R}^m$. Note that Taylor expansion defines a natural homomorphism

$$\chi \colon \mathrm{Diff}(\mathbf{R}^m, 0) \to \widehat{\mathrm{Diff}}(\mathbf{R}^m, 0)$$

which is not injective, but is surjective [31, Chap. I, p. 28]. We say that a group $G \subset \mathrm{Diff}(\mathbf{R}^m, 0)$ is *formally linearizable* if there exists $f \in \widehat{\mathrm{Diff}}(\mathbf{R}^m, 0)$ such that f conjugates $\chi(G)$ to its linear part.

Proof of Theorem 2.8. Suppose we have a C^∞-action

$$\Phi\colon SL(n, \mathbf{R}) \to \mathrm{Diff}(\mathbf{R}^m, 0).$$

Let $\phi = \chi \circ \Phi$ and let $D\colon SL(n, \mathbf{R}) \to GL(m, \mathbf{R})$ be the linear part of ϕ; that is D is the homomorphism: $D(g) = T^1\phi(g)$. The proof is an inductive argument. First set $h_1 = \mathrm{Id} \in \widehat{\mathrm{Diff}}(\mathbf{R}^m, 0)$. Then for some integer $l > 1$ suppose that one has $h_{l-1} \in \widehat{\mathrm{Diff}}(\mathbf{R}^m, 0)$ such that for each $g \in SL(n, \mathbf{R})$, the Taylor polynomial $T^{l-1}\left(h_{l-1}\phi(g)h_{l-1}^{-1}\right)$ is linear and equals $D(g)$; that is, setting $g_{l-1} = h_{l-1}\phi(g)h_{l-1}^{-1}$, one has $T^{l-1}(g_{l-1}) = D(g)$. Let $E_l(g)$ denote the homogeneous part of g_l of degree l. Clearly E_l is a function on $SL(n, \mathbf{R})$ with values in the space P_l of homogeneous polynomials of degree l with values in $\mathbf{R}^m$. In terms of the group operation in $SL(n, \mathbf{R})$, we have

$$E_l(gh) = E_l(g) \circ D(h) + D(g) \circ E_l(h). \tag{1}$$

Notice that $SL(n, \mathbf{R})$ acts linearly on P_l; explicitly, for each $p \in P_l$ and each $g \in SL(n, \mathbf{R})$, one sets

$$g \cdot p = D(g) \circ p \circ D(g^{-1}).$$

So we can consider the cohomology of $SL(n, \mathbf{R})$ with values in P_l, twisted by this action. Now let $c_l(g) = E_l(g) \circ D(g^{-1})$ and observe that from (1), c_l is a 1-cocycle; that is:

$$c_l(gh) = g \,.\, c_l(h) + c_l(g).$$

By Whitehead's lemma (see for example [14, Chapter VII.6]),

$$H^1\left(\mathfrak{sl}(n, \mathbf{R}), \mathbf{R}^m\right) = 0,$$

and hence by Van Est's Theorem [49], $H^1\left(SL(n, \mathbf{R}), \mathbf{R}^m\right) = 0$. So c_l is exact. Thus $c_l = dp_l$, for some $p_l \in P_l$; that is, $c_l(g) = g \cdot p_l - p_l$, for all $g \in SL(n, \mathbf{R})$. Consider the polynomial diffeomorphism $\eta = \mathrm{Id} + p_l \in \widehat{\mathrm{Diff}}(\mathbf{R}^m, 0)$. Note that

$$\eta^{-1} = \mathrm{Id} - p_l + \text{terms of order} > l.$$

Consider the conjugation of g_l by η. Modulo terms of order $> l$, one has:

$$\begin{aligned}
\eta g_l \eta^{-1} &\equiv (\mathrm{Id} + p_l) \circ \big(D(g) + E_l(g)\big) \circ (\mathrm{Id} - p_l) \\
&\equiv D(g) - D(g) \circ p_l + E_l(g) + p_l \circ D(g) \\
&\equiv D(g) - D(g) \circ p_l + c_l(g) \circ D(g) + p_l \circ D(g) \\
&\equiv D(g).
\end{aligned}$$

So, setting $h_l = \eta h_{l-1}$, we have that $T^l(h_l g h_l^{-1}) = D(g)$, for every $g \in SL(n,\mathbf{R})$. By induction, we have elements $h_l \in \widehat{\mathrm{Diff}}(\mathbf{R}^m,0)$ such that $T^l(h_l g h_l^{-1}) = D(g)$ for all $l > 0$. Finally set $h = \lim_{l\to\infty} h_l$. This makes sense in $\widehat{\mathrm{Diff}}(\mathbf{R}^m,0)$ and by construction, h formally linearizes the action Φ.

3. Preparatory results

First let us make some general comments:

REMARK 3.1. If a Lie group G acts on a topological manifold, then the restriction of the action to each orbit is a transitive G-action; that is, each orbit is a homogeneous space G/H for some closed subgroup $H \subset G$. In particular, transitive C^0-actions of $SL(n,\mathbf{R})$ are conjugate to analytic $SL(n,\mathbf{R})$-actions.

REMARK 3.2. Every non-trivial continuous action of $SL(n,\mathbf{R})$ is either faithful, or factors through a faithful action of $PSL(n,\mathbf{R})$. Indeed, not only is $SL(n,\mathbf{R})$ simple as a Lie group (that is, its proper normal subgroups are discrete), but when n is odd it is simple as an abstract group and when n is even $PSL(n,\mathbf{R}) = SL(n,\mathbf{R})/\{\pm 1\}$ is simple as an abstract group. In particular, if n is odd, every non-trivial continuous action of $SL(n,\mathbf{R})$ is faithful. If n is even, non-faithful $SL(n,\mathbf{R})$-actions are common: see, for example, the adjoint action of $SL(n,\mathbf{R})$ for n even, or the irreducible $SL(2,\mathbf{R})$-representation on $\mathbf{R}^{2p+1}$ (see Section 5).

REMARK 3.3. Every non-trivial C^1-action of $SL(n,\mathbf{R})$ on $(\mathbf{R}^n,0)$ is faithful. Indeed, the differential at the origin defines a homomorphism $D\colon SL(n,\mathbf{R}) \to GL(n,\mathbf{R})$. In fact, since $SL(n,\mathbf{R})$ is a simple Lie group, the image of D is contained in $SL(n,\mathbf{R})$. By Thurston's stability theorem, D can't be trivial. So, for dimension reasons, D maps onto $SL(n,\mathbf{R})$. If an $SL(n,\mathbf{R})$-action is not faithful, then by the previous Remark, n is even and the element -1 acts trivially. But then D defines a homomorphism from $PSL(n,\mathbf{R})$ onto $SL(n,\mathbf{R})$, which is impossible since $PSL(n,\mathbf{R})$ is simple.

REMARK 3.4. Suppose one has a C^1-action of $SL(n, \mathbf{R})$ on $(\mathbf{R}^n, 0)$. By the previous Remark, the differential D defines an automorphism of $SL(n, \mathbf{R})$. Let σ be the automorphism of $SL(n, \mathbf{R})$ defined by $\sigma(g) = (g^{-1})^t$, and let τ the automorphism given by conjugation by the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & \mathrm{Id}_{n-1} \end{pmatrix} \in GL(n, \mathbf{R}).$$

Recall (see [16, Theorem IX.5]) that the group of outer automorphisms of $SL(n, \mathbf{R})$ is generated by the involution σ if n is odd, and it is the group of order 4 generated by σ and τ if n is even — except when $n = 2$, in which case σ is the inner automorphism generated by conjugation by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Hence, up to conjugacy by an element of $GL(n, \mathbf{R})$, we may assume that the differential D is either the identity or the map σ.

Part (a) of the following theorem is classical (see [30, Chap. VI, Theorem 2]). Parts (b) and (c) could be deduced from Dynkin's classification of maximal subgroups of semi-simple Lie groups [8]; we give a more direct proof. We treat the case $n = 2$ of Part (c) in Section 6 below.

THEOREM 3.5.

(a) *There is no non-trivial C^0-action of $SL(n, \mathbf{R})$ on any topological manifold of dimension $m < n - 1$.*

(b) *Every non-trivial C^0-action of $SL(n, \mathbf{R})$ on an $(n-1)$-dimensional connected topological manifold is transitive and is conjugate to the projective action of $SL(n, \mathbf{R})$ on either S^{n-1} or $\mathbf{R}P^{n-1}$.*

(c) *For $n \geq 3$, every transitive C^0-action of $SL(n, \mathbf{R})$ on a non-compact n-dimensional topological manifold is conjugate, after possibly precomposing with some automorphism of $SL(n, \mathbf{R})$, to the canonical action of $SL(n, \mathbf{R})$ on $\mathbf{R}^n \backslash \{0\}$ or $(\mathbf{R}^n \backslash \{0\})/\{\pm \mathrm{Id}\} \cong \mathbf{R}P^{n-1} \times \mathbf{R}$.*

Proof. (a) Suppose that H is a closed subgroup of $SL(n, \mathbf{R})$ of codimension m. Consider the restricted $SO(n)$-action. Choose any Riemannian metric on the smooth manifold $M = SL(n, \mathbf{R})/H$ and average it by the $SO(n)$-action. Then $SO(n)$ acts isometrically, for the averaged metric. But the group of isometries of M has dimension at most $m(m+1)/2$, by [19, Theorem II.3.1]. So

$$\dim SO(n) = \binom{n}{2} \leq \binom{m+1}{2}.$$

Hence $n \leq m + 1$, as required.

(b) Suppose one has a non-trivial C^0-action of $SL(n, \mathbf{R})$ on an $(n-1)$-dimensional connected topological manifold M. By (a), this action is transitive and $M = G/H$ for some closed subgroup $H \subset G$. Then the restricted $SO(n)$-action gives a compact group of isometries of M of dimension $n(n-1)/2$. It follows from [19, Theorem II.3.1] that M is the round sphere S^{n-1}, or projective space $\mathbf{R}P^{n-1}$, and the action is the canonical one.

(c) Consider a transitive C^0-action of $SL(n, \mathbf{R})$ on an n-dimensional topological manifold M and let H denote the stabilizer of some point so that M can be identified with the homogeneous space $SL(n, \mathbf{R})/H$. We first deal with the case where H is connected, since the other cases can be reduced to this by taking a covering of the corresponding homogeneous space. We begin by showing that the linear action of $H \subset SL(n, \mathbf{R})$ on $\mathbf{R}^n$ is reducible and fixes a line or a hyperplane.

Suppose first by contradiction that the complexified representation of the Lie algebra $\mathfrak{H} \otimes \mathbf{C} \subset \mathfrak{sl}(n, \mathbf{C})$ is irreducible, where $\mathfrak{H}$ denotes the Lie algebra of H. By a well known theorem of Lie, the radical of $\mathfrak{H} \otimes \mathbf{C}$ preserves some line in $\mathbf{C}^n$ and since we assume that $\mathfrak{H} \otimes \mathbf{C}$ is irreducible, the only possibility is that this radical is Abelian and acts by homotheties. In other words, $\mathfrak{H} \otimes \mathbf{C}$ is a reductive algebra. By taking suitable real forms, one would have a compact subgroup K in $SU(n)$ whose real codimension is n. Now, as before, one can consider $SU(n)$ as a group of isometries of the n-dimensional manifold $SU(n)/K$. This would imply that $\dim SU(n) = n^2 - 1 \leq n(n-1)/2$ which is a contradiction.

On the other hand, if $\mathfrak{H} \otimes \mathbf{C} \subset \mathfrak{sl}(n, \mathbf{C})$ is a reducible representation, then $\mathfrak{H} \otimes \mathbf{C} \subset \mathfrak{sl}(n, \mathbf{C})$ is contained (up to conjugacy) in the algebra of matrices preserving $\mathbf{C}^p \times \{0\}$ (for some $0 < p < n$) which is of codimension $p(n-p)$. Therefore $p(n-p) \leq n$ so that $p = 1$ or $n-1$. This means that there is a complex line or a complex hyperplane fixed by $\mathfrak{H} \otimes \mathbf{C}$. This line or hyperplane has to be invariant under complex conjugation; otherwise we would have an invariant complex subspace of dimension or codimension 2 and this is not possible since H has codimension exactly n. It follows that H fixes a line or a hyperplane.

If H fixes a hyperplane, replace it by $\sigma(H)$ where σ is the automorphism of $SL(n, \mathbf{R})$ defined by $\sigma(g) = (g^{-1})^t$. This amounts to changing the action of $SL(n, \mathbf{R})$ under consideration by pre-composing with σ. So we can assume that H is contained in the stabilizer H' of the radial half-line Δ^+ through the first vector e_1 of the canonical basis in $\mathbf{R}^n$. Moreover, H is a codimension one subgroup of H'.

By Lie [23] (see also [33, Part II, Chap. 6, Theorem 2.1]), the connected codimension one closed subgroups of H' are given by homomorphisms ψ from H' to $\mathbf{R}$, $\mathbf{Aff}$, or (some cover of) $PSL(2,\mathbf{R})$, where

$$\mathbf{Aff} = \left\{ \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \quad : \quad a > 0 \right\}$$

is the group of affine transformations of the line. More precisely, H is (the component of the identity of) the inverse image by ψ of a codimension one subgroup, which is trivial in the case of $\mathbf{R}$, the subgroup of homotheties $(b = 0)$ in the case of $\mathbf{Aff}$ and the upper triangular subgroup in the case of $PSL(2,\mathbf{R})$. It is easy to see that there are no non-trivial homomorphisms of H' to $\mathbf{Aff}$. There are no non-trivial homomorphisms of H' to (any cover of) $PSL(2,\mathbf{R})$, except in the case $n = 3$. In this special case $n = 3$, one finds that H is the restricted upper-triangular group

$$U = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \quad : \quad a > 0 \right\},$$

which gives the compact flag manifold $SL(3,\mathbf{R})/U \cong S^3$. Finally, up to a multiplicative constant, there is a unique homomorphism from H' to $\mathbf{R}$:

$$\psi\colon (A_{ij}) \in H' \mapsto \ln A_{11} \in \mathbf{R}.$$

Note that here $H = \ker\psi$ is precisely the stabilizer $\mathrm{Stab}_{SL(n,\mathbf{R})}(e_1)$ of e_1 so that here $SL(n,\mathbf{R})/H$ is the homogeneous space $\mathbf{R}^n\backslash\{0\}$.

Thus we have dealt with the case where H is connected. Suppose that H is not connected, and let H_0 be its connected component of the identity. Now H_0 is a normal subgroup of H, and from above, by conjugation we may take H_0 to be either the group $\mathrm{Stab}_{SL(n,\mathbf{R})}(e_1)$, or the group U. If $H_0 = \mathrm{Stab}_{SL(n,\mathbf{R})}(e_1)$, notice that the normalizer of H_0 is the stabilizer H' of the radial half-line Δ^+. It follows that H/H_0 is a discrete subgroup of $\mathbf{R}$. If H/H_0 is finite, then $H/H_0 = \pm 1$ and so the quotient space is $\mathbf{R}^n\backslash\{0\}/\{\pm\mathrm{Id}\}$. If H/H_0 is infinite, then it is either infinite cyclic, or infinite cyclic cross $\mathbf{Z}/2\mathbf{Z}$, and in either case the quotient space is compact. If $H_0 = U$, the normalizer of H_0 is the full group $\overline{U}$ of upper-triangular matrices: there are 3 possibilities here, but in each case we get a compact quotient space.

This completes the proof of the theorem. □

We now describe a useful method of extending an action of a subgroup to an action of the larger group. This method is very general and variations of it appear in various branches of mathematics: “induced module” in representation theory, “suspension” in dynamical systems, etc. In particular, it was used in an essential way in Schneider's classification of analytic $SL(2,\mathbf{R})$-actions on surfaces [37]. Suppose that H is a closed subgroup of a Lie group G and suppose that H acts continuously on a topological space F. So H acts diagonally on $G \times F$, where $g \in H \subset G$ acts on the first factor by right translation by g^{-1}. Let $E = (G \times F)/H$ denote the quotient space. So E fibres over the space G/H of left cosets of H, with fibre F. Now notice that G acts on $G \times F$ by left translation on the first factor, and this defines an action of G on E.

DEFINITION 3.6. The action of G on E just described is called the *suspension of the action of H on F.*

Notice that for such an action, there is a H-invariant subspace F' in E, which is H-equivariantly homeomorphic to F, and which has the property that $\mathrm{Stab}_H(x) = \mathrm{Stab}_G(x)$, for all $x \in F'$. Indeed, one can take $F' = \pi^{-1}(H)$, where $\pi \colon E \to G/H$ is the natural fibration. Given $f \in F$ and $g \in G$, let $[g,f]$ denote the image in E of (g,f) under the quotient map $G \times F \to E$. Then $\pi[g,f] = gH$, and $F' = \{[1,f] : f \in F\}\big(SL(n,\mathbf{R})\big)$.

Conversely, one has:

LEMMA 3.7. *Let H be a closed subgroup of a Lie group G. Suppose that G acts continuously on a topological space M and that there is a G-equivariant fibration $p \colon M \to G/H$. Then the G-action on M is conjugate to the suspension of the action of H on the fibre $F = p^{-1}(H)$. More precisely, if $E = (G \times F)/H$, then there is a G-equivariant homeomorphism from M to E which projects to the identity map on G/H.*

Proof. We define a function $\psi \colon M \to E$ as follows: for each $x \in M$ we set

$$\psi(x) = [g, g^{-1}(x)],$$

where $p(x) = gH$. Note that this makes sense since $g^{-1}(x) \in F$ and the definition of $\psi(x)$ doesn't depend upon the choice of g. By construction, ψ is G-equivariant and projects to the identity map on G/H. Finally, it is easy to see that ψ is a homeomorphism. □

By Remark 2.2, $SO(n)$-actions of class C^0 on $(\mathbf{R}^m, 0)$ are not always linearizable. Despite this, we have the following result, which was proved for the cases $n \leq 3$ in [30, Chapter VI.6.5] and was conjectured therein for all n.

PROPOSITION 3.8. *Every faithful C^0-action of $SO(n)$ on $(\mathbf{R}^n, 0)$ is globally conjugate to the canonical linear action.*

Proof. By the proof of Theorem 3.5(a), the orbits of the $SO(n)$-action have dimension $\geq n-1$. In fact, there cannot be any $SO(n)$-orbit of dimension n, since otherwise it would be all of $\mathbf{R}^n \backslash \{0\}$, which is impossible, by the compactness of $SO(n)$. By the proof of Theorem 3.5(b), the only $SO(n)$-orbits of dimension $n-1$ are S^{n-1} and $\mathbf{R}P^{n-1}$, and the actions on them are conjugate to the canonical projective ones. In fact, for $n \geq 3$ there can be no orbit homeomorphic to $\mathbf{R}P^{n-1}$, because $\mathbf{R}P^{n-1}$ does not embed in $\mathbf{R}^n$ [6, Theorem 10.12]. So each orbit of $SO(n)$ is a $(n-1)$-dimensional sphere or a fixed point. It then follows from [30, ibid.] that 0 is the unique fixed point and there is a continuous ray γ beginning at 0 which meets each $SO(n)$-orbit exactly once.

First consider the $n=2$ case. Note that the $SO(2)$-action on $\mathbf{R}^2 \backslash \{0\}$ is free. Indeed, let $g \in SO(2)$ and suppose that $x \in \mathbf{R}^2 \backslash \{0\}$ belongs to the fixed point set $\mathrm{Fix}(g)$ of the action of g on $\mathbf{R}^2$. Then $\mathrm{Fix}(g)$ contains 0 as well as the entire orbit of x by $SO(2)$. By Eilenberg's theorem [9], since g is orientation preserving, the action of g on $\mathbf{R}^2$ is topologically conjugate to a rotation. So, as g has more than one fixed point, we must have $\mathrm{Fix}(g) = \mathbf{R}^2$. Hence, as the $SO(2)$-action on $\mathbf{R}^2$ is faithful by hypothesis, we have $g = \mathrm{Id}$, as claimed. Now define the map $\phi\colon \mathbf{R}^2 \to \mathbf{R}^2$ by setting

$$\phi\big(h\gamma(t)\big) = h \, . \begin{pmatrix} t \\ 0 \end{pmatrix},$$

for all $t \in [0, \infty)$, $h \in SO(2)$, where h acts on the left via the given $SO(2)$-action, and on the right by matrix multiplication. By construction, ϕ conjugates the given $SO(2)$-action to the canonical linear action.

Now suppose $n > 2$. Let $\{e_1, \dots, e_n\}$ denote the canonical basis of $\mathbf{R}^n$. Then, as in the proof in [30, ibid.], one may choose the ray γ to be comprised of fixed points of the restricted $SO(n-1)$-action, where here $SO(n-1)$ is the subgroup of $SO(n)$ which fixes the first basis vector e_1. So for each $x \in \mathbf{R}^n$, there is a unique number $t \in [0, \infty)$ and an element $g \in SO(n)$ such that $x = g\big(\gamma(t)\big)$. Moreover, for $x \in \mathbf{R}^n \backslash \{0\}$, the element g is unique modulo $SO(n-1)$. Consider the fibration

$$p\colon x \in \mathbf{R}^n \backslash \{0\} \mapsto g \in SO(n)/SO(n-1) \cong S^{n-1}.$$

Clearly p is $SO(n)$-equivariant. Notice that $p^{-1}\big(SO(n-1)\big) = \gamma \backslash \{0\} \cong \mathbf{R}$ and the $SO(n-1)$-action on this set is trivial. So, by Lemma 3.7, the action of $SO(n)$ on $\mathbf{R}^n \backslash \{0\}$ is conjugate to the action induced by the trivial action of $SO(n-1)$ on $\mathbf{R}$. That is, it is conjugate to the canonical action of $SO(n)$ on $\mathbf{R}^n \backslash \{0\}$. It remains to put back the origin. This can obviously be done equivariantly: one merely needs to verify that it can be done continuously. However, by averaging the flat metric on $\mathbf{R}^n$ by the original action of $SO(n)$, one may assume that the action is distance preserving. Thus, as t tends to 0, the $SO(n)$-orbits through $\gamma(t)$ converge uniformly to 0. So the continuity of the conjugation is clear. □

We will also need the following:

LEMMA 3.9. *Let $n \geq 3$ and suppose that one has a C^0-action of $SL(n, \mathbf{R})$ on $(\mathbf{R}^n, 0)$ such that the restricted action of $SO(n)$ is the canonical linear action. Then locally the $SL(n, \mathbf{R})$-action preserves the radial lines.*

Proof. The key point is that two points of $\mathbf{R}^n$ lie in the same radial line if and only if they have the same stabilizer under the $SO(n)$-action. Let $x, y \in \mathbf{R}^n$ lie in the same radial line and let $g \in SL(n, \mathbf{R})$. So $\mathrm{Stab}_{SO(n)}(x) = \mathrm{Stab}_{SO(n)}(y)$ and we want to show that

$$\mathrm{Stab}_{SO(n)}\big(g(x)\big) = \mathrm{Stab}_{SO(n)}\big(g(y)\big)\,.$$

Since the restricted action of $SO(n)$ is the canonical linear action, each orbit of $SL(n, \mathbf{R})$ in $\mathbf{R}^n \backslash \{0\}$ is either a round sphere centred at 0 or a spherical shell centred at 0. Suppose that our $SL(n, \mathbf{R})$-action on $\mathbf{R}^n$ has two spherical orbits, S_1 and S_2 say. By Theorem 3.5(b), the $SL(n, \mathbf{R})$-action on each sphere is the projective one. So there is an equivariant homeomorphism $\psi \colon S_1 \to S_2$. If $x \in S_1$ and $y = \psi(x) \in S_2$, we have $g(y) = \psi\big(g(x)\big)$ and as it is equivariant, ψ respects the stabilizers of the $SO(n)$-action. So $\mathrm{Stab}_{SO(n)}\big(g(y)\big) = \mathrm{Stab}_{SO(n)}\big(g(x)\big)$, as required (and ψ is just $\pm$ the radial projection of S_1 onto S_2).

By continuity, it remains to consider the case where x and y lie in the same open orbit of $SL(n, \mathbf{R})$; that is, suppose $y = h(x)$ for some $h \in SL(n, \mathbf{R})$. For all $f \in SL(n, \mathbf{R})$, one has $\mathrm{Stab}_{SO(n)}(x) = \mathrm{Stab}_{SO(n)}\big(f(x)\big)$ if and only if $f \in \mathrm{Norm}_{SL(n,\mathbf{R})}\big(\mathrm{Stab}_{SO(n)}(x)\big)$. So $h \in \mathrm{Norm}_{SL(n,\mathbf{R})}\big(\mathrm{Stab}_{SO(n)}(x)\big)$ and we need to show that $ghg^{-1} \in \mathrm{Norm}_{SL(n,\mathbf{R})}\big(\mathrm{Stab}_{SO(n)}\big(g(x)\big)\big)$. But if G is any group acting on a space X and H is a subgroup of G, then

$$\begin{aligned} g\big(\mathrm{Norm}_G\big(\mathrm{Stab}_H(x)\big)\big)g^{-1} &= \mathrm{Norm}_G\big(g\big(\mathrm{Stab}_H(x)g^{-1}\big)\big) \\ &= \mathrm{Norm}_G\big(\mathrm{Stab}_H\big(g(x)\big)\,, \end{aligned}$$

for all $x \in X$ and $g \in G$, as we require.

4. $SL(n,\mathbf{R})$-ACTIONS ON $\mathbf{R}^n$ FOR $n \geq 3$

Let $n \geq 3$. We first give examples of C^0-actions of $SL(n,\mathbf{R})$ on $\mathbf{R}^n$. Consider the canonical projective action of $SL(n,\mathbf{R})$ on S^{n-1}. Let Δ^+ be the radial half-line through the first basis element e_1 and let H denote the subgroup of $SL(n,\mathbf{R})$ that fixes Δ^+. So $SL(n,\mathbf{R})/H \cong S^{n-1}$. Consider the homomorphism

$$\psi\colon (A_{ij}) \in H \mapsto \ln A_{11} \in \mathbf{R}.$$

Notice that one obtains a linear action of H on $\mathbf{R}_*^+ = (0,\infty)$ by setting $h(x) = e^{\psi(h)}x$, for all $h \in H$, $x \in \mathbf{R}_*^+$. Obviously this is conjugate to the H-action on Δ^+. It follows from Lemma 3.7 that the action of $SL(n,\mathbf{R})$ obtained by suspension of this action of H on $\mathbf{R}_*^+$ is the canonical linear action of $SL(n,\mathbf{R})$ on $\mathbf{R}^n \backslash \{0\}$. In fact, the map

$$\psi\colon [g,x] \in \big(SL(n,\mathbf{R}) \times \mathbf{R}_*^+\big)/H \mapsto g(x e_1) \in \mathbf{R}^n\backslash\{0\}$$

is an isomorphism. We now deform the action of H. Choose a topological flow $(\varphi^t)_{t\in\mathbf{R}}$ on $\mathbf{R}^+ = [0,\infty)$, fixing 0. This defines an action of H on $\mathbf{R}_*^+$ by setting $h(x) = \varphi^{\psi(h)}(x)$, for all $h \in H$, $x \in \mathbf{R}_*^+$. Now suspend this action of H and let Φ denote the resulting action of $SL(n,\mathbf{R})$ on the space $M = \big(SL(n,\mathbf{R}) \times \mathbf{R}_*^+\big)/H$. The space M fibres over S^{n-1}, with fibre $\mathbf{R}_*^+$, and the structure group is orientation preserving. So topologically, M is $\mathbf{R}_*^+ \times S^{n-1}$. Thus, identifying $S^{n-1} \times \{0\}$ to a point, we obtain an $SL(n,\mathbf{R})$-action on $\mathbf{R}^n$. The fixed points of the flow φ correspond to orbits in $\mathbf{R}^n$ which are spheres of dimension $n-1$. In general, an n-dimensional orbit is either all of $\mathbf{R}^n\backslash\{0\}$, as in the linear case, or it is a spherical shell, bounded by S^{n-1} orbits, or a punctured ball bounded by an S^{n-1} orbit, or the exterior of an S^{n-1} orbit. In all cases, the n-dimensional orbits are conjugate to the canonical linear one on $\mathbf{R}^n\backslash\{0\}$, by Theorem 3.5(c).

THEOREM 4.1. *For all $n \geq 3$, every non-trivial C^0-action of $SL(n,\mathbf{R})$ on $(\mathbf{R}^n, 0)$ is conjugate to one of the above actions Φ.*

Proof. Suppose that we have a non-trivial C^0-action of $SL(n,\mathbf{R})$ on $(\mathbf{R}^n, 0)$. First use Proposition 3.8 to linearize the $SO(n)$-action. Then by Lemma 3.9, the $SL(n,\mathbf{R})$-action preserves the radial lines. Hence the radial projection $\mathbf{R}^n\backslash\{0\} \to S^{n-1}$ is equivariant, where the action of $SL(n,\mathbf{R})$ on S^{n-1} is the canonical projective one. Let H be the stabilizer of the radial half-line Δ^+ through e_1, as above. So the action of $SL(n,\mathbf{R})$ on $\mathbf{R}^n\backslash\{0\}$ is induced by some action of H on $\mathbf{R}$. Notice that this action is trivial when restricted to

$SO(n-1)$. It remains to consider all actions of H on $\mathbf{R}$ which are trivial on $SO(n-1)$. Again, by Lie [23, ibid.], these are given by homomorphisms from H to $\mathbf{R}$, $\mathbf{Aff}$, or (some cover of) $PSL(2,\mathbf{R})$. We have the homomorphism $\psi\colon (A_{ij}) \in H \mapsto \ln A_{11} \in \mathbf{R}$. Note that $\ker\psi = SL(n-1,\mathbf{R}) \ltimes \mathbf{R}^{n-1}$. But it is easy to see that there are no non-trivial homomorphisms of $\ker\psi$ to $\mathbf{R}$ or $\mathbf{Aff}$. There are no non-trivial homomorphisms of $\ker\psi$ to $SL(2,\mathbf{R})$, except in the case $n=3$, and in this case there are no such homomorphisms which are trivial on $SO(n-1)$. So the only possibility left is that H acts on $\mathbf{R}$ by some flow. Finally, we put back the origin, as in the proof of Proposition 3.8. This completes the proof of the theorem. □

We now prove Theorem 1.1 for $n \geq 3$.

THEOREM 4.2. *For all $n \geq 3$ and $k = 1,\ldots,\infty$, every C^k-action of $SL(n,\mathbf{R})$ on $(\mathbf{R}^n, 0)$ is C^k-linearizable.*

Proof. Let $n \geq 3$ and $k = 1,\ldots,\infty$ and suppose that we have a non-trivial C^k-action of $SL(n,\mathbf{R})$ on $(\mathbf{R}^n,0)$. By Remark 3.4, we may assume that the differential of the action at the origin is either the identity or the map $g \mapsto (g^{-1})^t$. We will assume that it is the identity; the other possibility can be handled using the same argument.

Linearizing the $SO(n)$-action, using the Bochner-Cartan theorem, one may assume that the $SO(n)$-action is the canonical one. Then by Lemma 3.9, the $SL(n,\mathbf{R})$-action preserves the radial lines. Let Δ denote the radial line through the first of the canonical basis elements, e_1. Consider $H = \mathrm{Stab}_{SL(n,\mathbf{R})}(\Delta)$, as before. So, as we saw in the proof of Theorem 4.1, H defines a C^k-flow on Δ. This flow is hyperbolic, by the first paragraph. Hence by Theorem 2.5, this flow is linearizable by some local C^k-diffeomorphism f of $\Delta(\cong \mathbf{R})$. So, after conjugacy, we may assume that H acts linearly on Δ. Now define the local C^k-diffeomorphism F of $\mathbf{R}^n$ by the formula:

$$F(x) = \begin{cases} \frac{f(\|x\|)}{\|x\|}\, x, & x \neq 0 \\ 0, & x = 0\,. \end{cases} \tag{2}$$

To see that F is of class C^k, the key point is to verify that f is a C^k *odd* function on $\mathbf{R}$. This follows easily from the fact that the flow on Δ commutes with $\mathrm{Stab}_{SO(n,\mathbf{R})}(\Delta)$, and the $SO(n)$-action is linear.

Now notice that F agrees with f on $\Delta^+ = \{t e_1 \in \Delta \;:\; t \geq 0\}$, and as F commutes with the $SO(n)$-action, the $SO(n)$-action is unchanged by conjugation by F. In particular, the $SO(n)$-action still commutes with dilations.

It follows that after conjugation by F, the $SL(n,\mathbf{R})$-action commutes with dilations. Indeed, consider the conjugated $SL(n,\mathbf{R})$-action. If $f \in SL(n,\mathbf{R})$, $x \in \mathbf{R}^n$ and $\lambda > 0$, then choose $a, b \in SO(n)$ such that $ax \in \Delta^+$ and $bf(\lambda x) \in \Delta^+$. Provided x is sufficiently close to 0, ax and $bf(\lambda x)$ will lie in the domain of f. Then $bfa^{-1} \in H$ and so

$$\begin{aligned} f(\lambda x) &= b^{-1} bfa^{-1} a(\lambda x) = b^{-1}(bfg^{-1})\,\lambda a(x) \\ &= b^{-1}\lambda(bfa^{-1})\,a(x) = \lambda b^{-1}(bfa^{-1})\,a(x) \\ &= \lambda f(x)\,. \end{aligned}$$

The proof of the theorem is then completed by the following well known result (cf. [17, Lemma 2.1.4]). □

LEMMA 4.3. *Every C^1 map commuting with dilations is linear.*

Proof. Suppose that f is a C^1-diffeomorphism of $\mathbf{R}^n$ which commutes with dilations. By comparing the differential of $\lambda . f$ and $f \circ \lambda$ at x we have $\lambda df|_x = \lambda df|_{\lambda x}$, for each $\lambda > 0$ and every $x \in \mathbf{R}^n$. Hence $df|_x = df|_{\lambda x}$ and so df is constant on the radial lines. Thus $df|_x = df|_0$ for all x and so f is linear. □

5. THE ADJOINT REPRESENTATION OF $SL(2,\mathbf{R})$

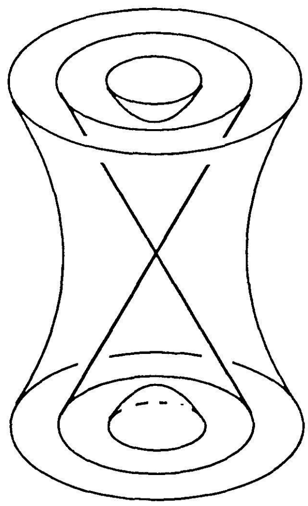
Let us recall some facts concerning the linear representations of $SL(2,\mathbf{R})$. Let $P_l(\mathbf{R}^2)$ denote the space of real valued homogeneous polynomials, of two variables, of degree l. As a vector space, $P_l(\mathbf{R}^2) \cong \mathbf{R}^{l+1}$, and the action of $SL(2,\mathbf{R})$ on $\mathbf{R}^2$ defines a linear action on $P_l(\mathbf{R}^2)$: up to isomorphism, this is the (unique) irreducible representation of $SL(2,\mathbf{R})$ in dimension $l+1$. In dimension 3, there is another useful realization of the polynomial representation, called the *adjoint representation*. Notice that the group $SL(2,\mathbf{R})$ acts by the adjoint representation on its Lie algebra $\mathfrak{sl}(2,\mathbf{R})$. Of course, $\mathfrak{sl}(2,\mathbf{R})$ is the space of 2×2 real traceless matrices; so as a vector space, $\mathfrak{sl}(2,\mathbf{R}) \cong \mathbf{R}^3$. The adjoint representation $Ad\colon SL(2,\mathbf{R}) \to GL(3,\mathbf{R})$, defined by

$$Ad(g)\colon h \mapsto ghg^{-1}, \quad \forall g \in SL(2,\mathbf{R}),\ h \in \mathfrak{sl}(2,\mathbf{R}),$$

is an irreducible linear representation. In fact, an explicit equivariant isomorphism $\psi\colon \mathfrak{sl}(2,\mathbf{R}) \to P_2(\mathbf{R}^2)$ is obtained by taking $\psi(h)$, as a function of variables x and y, to be the area of the parallelogram spanned by (x,y) and $h(x,y)$. That is,

$$\psi\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = by^2 + 2axy - cx^2 .$$

Recall that the Cartan-Killing form K of a semi-simple Lie algebra is a non-degenerate quadratic form which is invariant under the adjoint representation of the associated Lie group. For $\mathfrak{sl}(2,\mathbf{R})$, one has $K = -8\det$. (The Cartan-Killing form is unique up to constant factor: the factor here of -8 corresponds to the usual convention $K = tr\, Ad^2$.) Notice that in particular, K has signature $(-,+,+)$ and hence determines a Minkowski metric on $\mathfrak{sl}(2,\mathbf{R})$. The time-like elements $h \in \mathfrak{sl}(2,\mathbf{R})$ (those with $\det h > 0$) are called *elliptic* elements. The space-like, resp. light-like, elements (that is, those with $\det h < 0$, resp. $\det h = 0$) are said to be *hyperbolic*, resp. *parabolic*. Notice that under ψ, the elliptic elements correspond to quadratics which are irreducible over $\mathbf{R}$, the hyperbolic elements correspond to products of distinct linear factors, and the parabolic elements correspond to (± 1 times) the squares of linear factors. Moreover, this equips Minkowski space with a "temporal" orientation: the parabolic elements which are squares of linear factors belong to the *future*.



Orbits of the adjoint representation

We denote the exponential map by $\exp\colon \mathfrak{sl}(2,\mathbf{R}) \to SL(2,\mathbf{R})$. It is common to say that $g = \exp h$ is parabolic, resp. elliptic, resp. hyperbolic, according to the type of h. The parabolic elements $g \in SL(2,\mathbf{R})$ are those with $tr^2(g) = 4$, the elliptic elements have $tr^2(g) < 4$, and the hyperbolic elements have $tr^2(g) > 4$. Notice also that the universal cover $\widetilde{SL}(2,\mathbf{R})$ of $SL(2,\mathbf{R})$ is also a Lie group. Let us denote the corresponding exponential map by $\widetilde{\exp}\colon \mathfrak{sl}(2,\mathbf{R}) \to \widetilde{SL}(2,\mathbf{R})$. The kernel of the natural quotient map $\widetilde{SL}(2,\mathbf{R}) \to SL(2,\mathbf{R})$ is precisely the image under $\widetilde{\exp}$ of the elliptic elements

$$\left\{ 2\pi n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \; : \; n \in \mathbf{Z} \right\} .$$

Now consider the orbits of the points $h \in \mathfrak{sl}(2, \mathbf{R})$ under the adjoint representation of $SL(2, \mathbf{R})$. Notice that since this action leaves K invariant, the action preserves the spheres $K =$ constant, in Minkowski space $\mathbf{R}^{1,2}$. (Of course, these Minkowski "spheres" are hyperboloids of revolution in $\mathbf{R}^3$. See figure.) So the orbits of the adjoint representation lie in these Minkowski spheres. In fact, it is easy to see that the orbits are precisely the connected components of these Minkowski spheres. (This is essentially the Jordan canonical form theorem in dimension 2.) In the case of non-zero parabolic elements, this means that the orbits are precisely the connected components of the light-cone minus the origin. Typical stabilizers of the adjoint representation are:

$$\text{hyperbolic case:} \quad \mathrm{Stab}_{SL(2,\mathbf{R})}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \left\{ \pm \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbf{R} \right\}$$

$$\text{parabolic case:} \quad \mathrm{Stab}_{SL(2,\mathbf{R})}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \left\{ \pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbf{R} \right\}$$

$$\text{elliptic case:} \quad \mathrm{Stab}_{SL(2,\mathbf{R})}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} : t \in \mathbf{R} \right\}.$$

For every non-zero element $h \in \mathfrak{sl}(2, \mathbf{R})$, the stabilizer $\mathrm{Stab}_{SL(2,\mathbf{R})}(h)$ is (± 1 times) the one-parameter subgroup $\{\exp(th) : t \in \mathbf{R}\}$ generated by h. Notice that if $h \in \mathfrak{sl}(2, \mathbf{R})$ is elliptic (resp. hyperbolic or parabolic), then $\mathrm{Stab}_{SL(2,\mathbf{R})}(h)$ is a circle (resp. two lines).

6. $SL(2, \mathbf{R})$-ACTIONS ON $\mathbf{R}^2$

By Theorem 3.5, the only homogeneous space of $SL(2, \mathbf{R})$ of dimension 1 on which $SL(2, \mathbf{R})$ acts faithfully is the circle S^1 equipped with the projective action. We now examine the homogeneous spaces of $SL(2, \mathbf{R})$ of dimension 2.

LEMMA 6.1. *Every faithful transitive action of $SL(2, \mathbf{R})$ on a noncompact surface is conjugate to one of the following two actions:*

(*a*) *the canonical action on* $SL(2, \mathbf{R}) / \left\{ \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} : t > 0 \right\} \cong \mathbf{R}^2 \backslash \{0\}$,

(*b*) *the canonical action on* $SL(2, \mathbf{R}) / \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbf{R} \right\} \cong \mathbf{R}^2 \backslash \{0\}$.

Proof. Of course, the homogeneous spaces of $SL(2, \mathbf{R})$ of dimension 2 are determined by the closed subgroups of $SL(2, \mathbf{R})$ of dimension 1. The connected component of a closed subgroup of $SL(2,\mathbf{R})$ of dimension 1 is a one-parameter subgroup: so it is either hyperbolic parabolic, or elliptic. This gives the following three homogeneous spaces:

(a) $$SL(2,\mathbf{R})/\left\{\begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} : t>0\right\} \cong \mathbf{R}^2\backslash\{0\},$$

(b) $$SL(2,\mathbf{R})/\left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t\in\mathbf{R}\right\} \cong \mathbf{R}^2\backslash\{0\},$$

(c) $$SL(2,\mathbf{R})/\left\{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta\in\mathbf{R}\right\} \cong \mathbf{R}^2.$$

Up to a twofold covering, these actions are just the restrictions of the adjoint representation to the orbits seen in the previous section. Notice however that in the elliptic case the element $-\mathrm{Id}$ acts trivially, and so the action is not faithful. So this leaves the two required actions.

It remains to show that the homogeneous spaces of the form $SL(2,\mathbf{R})/H$, where H is *not* connected, do not give us any new faithful actions. But it is easy to see that in the hyperbolic case, there are only two possibilities, corresponding to H having 2 or 4 connected components, and $-\mathrm{Id}$ acts trivially in each case. In the parabolic case, the situation is similar to that of Part (c) of Theorem 3.5: either $-\mathrm{Id}$ acts trivially, or the homogeneous space is compact. □

We now classify the continuous $SL(2,\mathbf{R})$-actions on $\mathbf{R}^2$. As in the higher dimensional case, we do this by giving a recipe for constructing examples, and then prove that this gives a complete list.

First, consider the oriented annulus $A=\{(r,\theta) : 1/2<r<2\}$, expressed in polar coordinates. Note that the above lemma furnishes us with three faithful transitive actions of $SL(2,\mathbf{R})$ on A. By conjugation by the map $\psi\colon \mathbf{R}^2\backslash\{0\}\to A$ defined by $\psi(r,\theta)=\left(\frac{1+2r}{2+r},\theta\right)$, the action (b) on $\mathbf{R}^2\backslash\{0\}$ gives us an action on A which we denote $\mathcal{P}^+$. By conjugating this by the inversion $(r,\theta)\mapsto(1/r,\theta)$, we obtain another action, which we denote $\mathcal{P}^-$. In the hyperbolic case (a), the above lemma gives us another action, which we denote $\mathcal{H}$, but it is easy to see that in this case, inversion gives us an isomorphic action.

Now choose a closed set $S\subset\mathbf{R}_*^+$ and choose a continuous function $T\colon \mathbf{R}_*^+\backslash S\to\{-1,0,1\}$. Then one obtains an $SL(2,\mathbf{R})$-action $\Phi_{S,T}$ on $(\mathbf{R}^2,0)$ as follows: taking $\mathbf{R}_*^+$ to be the radial coordinate, for each $s\in S$ one takes the circle of radius s to be a one-dimensional orbit, equipped with the canonical projective action, and for each connected component C of $\mathbf{R}_*^+ \backslash S$, one takes an action $\mathcal{P}^+, \mathcal{P}^-$ or $\mathcal{H}$ according to whether $T(C)$ is $1, -1$ or 0 respectively. It is easy to see that the actions on the two-dimensional orbits agree on their boundaries with the action on the one-dimensional orbit, so one does indeed obtain a continuous action.

THEOREM 6.2. *Every faithful C^0-action of $SL(2,\mathbf{R})$ on $(\mathbf{R}^2, 0)$ is conjugate to one of the above actions $\Phi_{S,T}$.*

Proof. First we linearize the $SO(2)$-action, using Proposition 3.8. This shows that the origin is the only zero-dimensional orbit, and that the one-dimensional orbits are circles centred at the origin. Moreover, from above, the restricted $SL(2,\mathbf{R})$-action on the one-dimensional orbits is the canonical projective action, and the actions on the two-dimensional orbits are each individually conjugate to either $\mathcal{P}^+, \mathcal{P}^-$ or $\mathcal{H}$. It remains to see that the open orbits can be glued to their boundaries in a unique manner.

Notice that if x lies in a one-dimensional orbit Ω, then $\mathrm{Stab}_{SL(2,\mathbf{R})}(x)$ contains a unique one-parameter parabolic subgroup G_x of $SL(2,\mathbf{R})$, and conversely, each one-parameter parabolic subgroup G_x fixes a unique pair of points $\pm x \in \Omega$. Inside the orbits of $\mathcal{P}^+$ and $\mathcal{P}^-$, the fixed point sets of the subgroups G_x are radial lines passing from one boundary component of the annulus to the other component. It follows that each end can be glued to a circle in precisely two ways which respect the action of the one-parameter parabolic subgroups. In fact, since $-\mathrm{Id}$ commutes with the $SL(2,\mathbf{R})$-action, the resulting actions are isomorphic.

Similarly, one treats the hyperbolic two-dimensional orbit of $\mathcal{H}$ by considering the fixed points sets of the one-parameter hyperbolic subgroups of $SL(2,\mathbf{R})$. If Ω is a one-dimensional orbit, then each one-parameter hyperbolic subgroup fixes four points in Ω. Conversely, each point $x \in \Omega$ is fixed by a family F_x of one-parameter hyperbolic subgroups. For the action $\mathcal{H}$, the one-parameter hyperbolic subgroups are the stabilizers of the points, and each one-parameter hyperbolic subgroup has precisely four fixed points. For each $x \in \Omega$, the fixed points of the elements of F_x define four curves which pass from one boundary component of the annulus to the other. It is not difficult to see that a unique $SL(2,\mathbf{R})$-action results by gluing each end of the annulus to a circle in such a way as to have continuity of these fixed point sets. □

We now complete the proof of Theorem 1.1.

THEOREM 6.3. *For all* $k = 1, \ldots, \infty$, *every* C^k*-action of* $SL(2,\mathbf{R})$ *on* $(\mathbf{R}^2, 0)$ *is* C^k*-linearizable.*

Proof. The proof is essentially the same as that of Theorem 4.2, except that we require a replacement for Lemma 3.9. Of course, it is not true that two points of $\mathbf{R}^2$ lie in the same radial line if and only if they have the same stabilizer under the $SO(2)$-action. The idea is to instead use the stable manifolds of the hyperbolic elements of $SL(2,\mathbf{R})$.

Let $\Phi\colon SL(2,\mathbf{R}) \to \mathrm{Diff}(\mathbf{R}^2, 0)$ be our given C^1-action. First note that as in the proof of Theorem 4.2, we may assume that locally the $SO(2)$-action is the canonical linear one and that the differential of Φ at the origin is the identity. Now let

$$h^t = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}$$

and consider the hyperbolic flow $\phi^t = \Phi(h^t)$ on $(\mathbf{R}^2, 0)$. By the stable manifold theorem (see [17, Theorem 6.2.8 and Theorem 17.4.3]), the stable manifold S_0 of ϕ^t is locally the graph of a C^1-function from $(\mathbf{R}, 0)$ to $(\mathbf{R}, 0)$. It follows that there is a local C^1-diffeomorphism of $(\mathbf{R}^2, 0)$ which commutes with the $SO(2)$-action and which takes S_0 to the x-axis. Conjugating Φ by this diffeomorphism, we may assume that locally S_0 is the x-axis. Then by using Theorem 2.5 we may linearize the action of ϕ^t on S_0, with some local C^k-diffeomorphism f of the x-axis and then extend the conjugation to $(\mathbf{R}^2, 0)$, using Equation (2) of Section 4. The upshot of this is that we may assume that, at least locally, the $SO(2)$-action is the canonical one, and the action of the subgroup $H = \{h^t : t \in \mathbf{R}\} \subset SL(2,\mathbf{R})$ is linear on the x-axis.

We will show that the $SL(2,\mathbf{R})$-action now preserves the radial lines. Let $R_\theta \in SO(2)$ denote the rotation through angle θ and let $f^t_\theta = R_\theta\, h^t R_\theta^{-1}$. Then clearly the stable manifold S_θ of $\Phi(f^t_\theta)$ is the radial line at angle θ. Now let $g \in SL(2,\mathbf{R})$ and consider $\Sigma = \Phi(g)(S_\theta)$. We want to show that Σ is a radial line. Clearly Σ is the stable manifold of the hyperbolic flow $\Phi(g f^t_\theta g^{-1})$. Let σ denote the angle of the stable line of the hyperbolic one-parameter group of matrices $g f^t_\theta g^{-1}$. Then $R_\sigma^{-1}\Sigma$ is the stable manifold Σ_A of the hyperbolic flow $\Phi(A^t)$, where $A^t = R_\sigma^{-1}\, g f^t_\theta g^{-1}\, R_\sigma$. Now the stable line of the hyperbolic flow A^t is the x-axis; that is A^t is a one-parameter subgroup of the form

$$A^t = \exp(t \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}) = \begin{pmatrix} e^{-t} & -b \sinh t \\ 0 & e^t \end{pmatrix}$$

for some $b \in \mathbf{R}$. We are required to show that Σ_A is the x-axis. First notice that *restricted to the x-axis*, one has

$$\Phi(A^t) = \Phi\left(\begin{pmatrix} 1 & b(e^{-2t}-1)/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}\right)$$
$$= \Phi\left(\begin{pmatrix} 1 & b(e^{-2t}-1)/2 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}$$

since H acts linearly on the x-axis. Hence, since the family of maps

$$F_t = \Phi\left(\begin{pmatrix} 1 & b(e^{-2t}-1)/2 \\ 0 & 1 \end{pmatrix}\right) \qquad t \geq 0$$

is equicontinuous in some neighbourhood of the identity, we conclude that Σ_A is the x-axis, as required.

By the above argument, we may assume that locally the $SO(2)$-action is the canonical one and the $SL(2,\mathbf{R})$-action preserves the radial lines. The proof is then completed as in the proof of Theorem 4.2. □

7. EXAMPLES OF C^0-ACTIONS OF $SL(2,\mathbf{R})$ ON $\mathbf{R}^m$

When m is greater than n there is a plethora of examples of continuous actions of $SL(n,\mathbf{R})$ on $(\mathbf{R}^m, 0)$. In this section we give some examples in the case $n = 2$.

7.1. THE SYMMETRIC PRODUCT. Choose one of the continuous $SL(2,\mathbf{R})$-actions on $(\mathbf{R}^2, 0)$ from the previous section. Now consider the associated $SL(2,\mathbf{R})$-action on the symmetric product

$$\Pi_{i=1}^m \mathbf{R}^2/\Sigma_m \cong \mathbf{C}^m,$$

where Σ_m is the symmetric group on m letters. Recall that the last identification associates to an m-tuple of points $(x_1, \dots, x_m)$ in $\mathbf{R}^2 \cong \mathbf{C}$ the coefficients of the monic polynomial of degree m in one complex variable whose roots are the x_i. As the original action fixed the origin in $\mathbf{R}^2$, so the corresponding action fixes the origin in $\mathbf{R}^{2m}$.

7.2. THE ADJOINT ACTION AT INFINITY. Consider the adjoint action of $SL(2,\mathbf{R})$ on $\mathbf{R}^3$, as discussed in Section 5. Removing the origin and compactifying the other end, we obtain a C^0-action of $SL(2,\mathbf{R})$ on $\mathbf{R}^3$, which we will call the *adjoint action at infinity*. This action is certainly not topologically linearizable, since all the orbits now accumulate to the fixed point. In fact, this action is not topologically conjugate to any C^1-action. To see this, consider the hyperbolic element $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Using the exponential

$\exp(th)$, one obtains a one-parameter subgroup in $SL(2,\mathbf{R})$ which, by the adjoint action, defines a flow $\mathfrak{F}$ on $\mathfrak{sl}(2,\mathbf{R})$. Choose the following basis for $\mathfrak{sl}(2,\mathbf{R}) \cong \mathbf{R}^3$:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

Then a simple computation shows that the flow $\mathfrak{F}$ is generated by the vector field $X = z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$ (where (x,y,z) are the coordinates with respect to the above basis). Restricted to each plane $x = constant$, the vector field X has a standard hyperbolic singularity, with index -1, and on the invariant lines $z = -y$ and $z = y$, the flow is contracting and expanding respectively. It follows that if the $SL(2,\mathbf{R})$-action at infinity was C^1, then the differential at infinity of the action of X would be trivial. In this case, the differential at infinity of the entire $SL(2,\mathbf{R})$-action would be trivial, contradicting Thurston's stability theorem.

7.3. THE ACTION ON THE CLOSED SUBGROUPS OF $\mathbf{R}^2$. Recall that from [35] the space Gr of closed subgroups of $\mathbf{R}^2$, with the Hausdorff topology, is homeomorphic to S^4. Obviously $SL(2,\mathbf{R})$ acts continuously on Gr, and the two trivial subgroups, $\{0\}$ and $\mathbf{R}^2$, are fixed by this action. Inside Gr there is an invariant S^3 comprised of the set K of subgroups isomorphic to $\mathbf{R}$, together with the set of subgroups isomorphic to $\mathbf{Z}^2$ which have generators which span a parallelogram of area 1. The set K, which is a trefoil knot in S^3, is a 1-dimensional orbit, and its complement $S^3 - K$ is a single 3-dimensional orbit.

Removing one of the fixed subgroups, $\{0\}$ or $\mathbf{R}^2$, one obtains an interesting $SL(2,\mathbf{R})$-action on $\mathbf{R}^4$ with one fixed point. Notice that this action is not conjugate to a C^1-action. Indeed, if the action was C^1, then the differential at the origin would define a linear representation of $SL(2,\mathbf{R})$ in $\mathbf{R}^4$. So this representation would be a direct sum of irreducible representations. Since $-\mathrm{Id}$ acts trivially on Gr, it follows that it is either the sum of the canonical 3-dimensional representation with the trivial 1-dimensional representation, or it is the trivial 4-dimensional representation. But the second case is not possible, by Thurston's stability theorem. In the first case, one could linearize the $SO(2)$-action, using the Bochner-Cartan theorem, and thus locally one would find a 2-dimensional subspace through the origin which was fixed pointwise by $SO(2)$. But there are no closed subgroups of $\mathbf{R}^2$ which are $SO(2)$-invariant, apart from $\{0\}$ and $\mathbf{R}^2$. So this case is also impossible.

7.4. CONING ACTIONS ON SPHERES. If one has a non-trivial $SL(2,\mathbf{R})$-action on S^m, then taking the cone in the obvious sense, one obtains an $SL(2,\mathbf{R})$-action on $(\mathbf{R}^{m+1},0)$. We claim that such actions cannot be conjugate to C^1 actions. Indeed, actions defined by coning have invariant spheres around 0. If a C^1 diffeomorphism has a family of invariant topological spheres around the origin, it cannot have any stable manifold so that all the eigenvalues of its differential at the origin have modulus one. No non-trivial linear representation of $SL(2,\mathbf{R})$ has the property that all eigenvalues of all elements have modulus one. So, if the action under consideration was C^1 the differential at the origin would be trivial: this is a contradiction with Thurston's stability theorem.

There are many interesting actions of $SL(2,\mathbf{R})$ on spheres. Compactifying the actions of Section 6 gives examples on S^2. An action on S^3 was given in Example 7.3. Notice also that if one has actions of $SL(2,\mathbf{R})$ on S^p and S^q, then there is an associated action of $SL(2,\mathbf{R})$ on their join $S^p * S^q = S^{p+q+1}$.

Finally we remark that many interesting actions of $SL(n,\mathbf{R})$ on spheres, for $n \geq 3$, can be found in the papers of Fuichi Uchida (see for example [46, 47, 48]).

8. A C^∞-ACTION OF $SL(2,\mathbf{R})$ WHICH IS NOT LINEARIZABLE

Here we give a variation of the Guillemin-Sternberg example a C^∞-action of the Lie algebra $\mathfrak{sl}(2,\mathbf{R})$ on $\mathbf{R}^3$ which is not linearizable. The action we give below integrates to a C^∞ non-linearizable $SL(2,\mathbf{R})$-action. It is obtained by deforming the adjoint action of $SL(2,\mathbf{R})$ on its Lie algebra. The constructed action is clearly non-linearizable since it has an orbit of dimension 3.

By differentiation, the adjoint action of $SL(2,\mathbf{R})$ defines a Lie algebra $\mathfrak{g}$ (isomorphic to $\mathfrak{sl}(2,\mathbf{R})$) of vector fields on $\mathbf{R}^3$. This algebra can be explicitly computed as follows: choose an element $h \in \mathfrak{sl}(2,\mathbf{R})$, take its exponential $\exp h$, and compute the derivative of the adjoint map $Ad\big(\exp(th)\big)$ at $t=0$. A convenient basis for $\mathfrak{g}$ is:

$$X = z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}, \quad Y = z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}, \quad R = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Here R is the derivative of $Ad\big(\exp(th)\big)$ where $h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The commutator relations are:

$$[X,Y] = -R, \quad [R,X] = Y, \quad [R,Y] = -X.$$

The idea is now to deform this action by adding in a component in the direction of the radial vector field:

$$\mathbf{r} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\,.$$

We don't change R, but we set $\bar{X} = X + f\mathbf{r}, \bar{Y} = Y + g\mathbf{r}$, for some functions f and g and we want to impose the same relations as before:

$$[\bar{X}, \bar{Y}] = -R, \quad [R, \bar{X}] = \bar{Y}, \quad [R, \bar{Y}] = -\bar{X}\,.$$

Since $\mathbf{r}$ commutes with X, Y and R, this requires

(3) $$R(f) = g$$

(4) $$R(g) = -f$$

(5) $$X(g) - Y(f) + f\mathfrak{r}(g) - g\mathfrak{r}(f) = 0.$$

Equations (3) and (4) give $R^2 f + f = 0$, which suggests that one looks for functions of the form

$$f(x, y, z) = xA(z, \sqrt{x^2 + y^2})\,,$$

for some function $A\colon \mathbf{R}^2 \to \mathbf{R}$. Then $g(x, y, z) = -yA(z, \sqrt{x^2 + y^2})$ and equation (5) gives

$$X(yA) + Y(xA) = 0\,.$$

This has the smooth solution

$$A(z, v) = \frac{a(v^2 - z^2)}{v^2}\,,$$

where $a\colon \mathbf{R} \to \mathbf{R}$ is any C^∞-function which is zero on $\mathbf{R}^-$. It follows that for each choice of a, the vector fields

$$\bar{X} = X + xA(z, \sqrt{x^2 + y^2})\mathfrak{r}, \quad \bar{Y} = Y - yA(z, \sqrt{x^2 + y^2})\mathfrak{r}, \quad \text{and} \quad R$$

generate a Lie algebra $\mathfrak{A}$ isomorphic to $\mathfrak{sl}(2, \mathbf{R})$. By choosing a to be a bounded function, we guarantee that the elements of $\mathfrak{A}$ are complete vector fields. Indeed, take a Riemannian metric on $\mathbf{R}^3$ which, outside the unit ball, is $g_{(x,y,z)}/\sqrt{x^2 + y^2 + z^2}$, where $g_{(x,y,z)}$ denotes the standard Euclidean metric. This defines a complete Riemannian metric, with respect to which the elements of $\mathfrak{A}$ are bounded. Hence, by [1, Proposition 2.1.21] for example, the elements of $\mathfrak{A}$ are complete. By integration, we consequently obtain a smooth action of the universal cover of $SL(2, \mathbf{R})$ whose orbits are those of $\mathfrak{A}$. In fact, since we haven't changed the definition of R, this gives a smooth action of $SL(2, \mathbf{R})$ whose orbits are those of $\mathfrak{A}$.

Notice that the vector fields $\overline{X}, \overline{Y}, R$ are linearly independent wherever $a \neq 0$. Indeed, putting $v = \sqrt{x^2 + y^2}$, one has:

$$\det(\overline{X}, \overline{Y}, R) = \det \begin{pmatrix} x^2 A(z,v) & z + xyA(z,v) & y + xzA(z,v) \\ z - xyA(z,v) & -y^2 A(z,v) & x - yzA(z,v) \\ -y & x & 0 \end{pmatrix}$$

$$= -(v^2 - z^2)\, v^2 A(z,v) = -(v^2 - z^2)\, a(v^2 - z^2).$$

It follows that if the function a is non-zero on $\mathbf{R}^+$, then the set of hyperbolic points in $\mathfrak{sl}(2, \mathbf{R})$ constitute a single orbit under the new action of $SL(2, \mathbf{R})$. Since no linear action of $SL(2, \mathbf{R})$ in $\mathbf{R}^3$ has an orbit of dimension 3, we conclude that our new action of $SL(2, \mathbf{R})$ is not linearizable. Note that outside the open orbit, this action coincides with the adjoint linear action.

In order to motivate the construction that we shall present in the next section, we now present another way of describing the non-linearizable action that we just constructed. Consider the subgroup Diag of $SL(2, \mathbf{R})$ of diagonal matrices and consider the trivial action of Diag on the positive line $\mathbf{R}^+_*$. It is easy to see that the suspension of this action is conjugate to the adjoint action of $SL(2, \mathbf{R})$ outside the invariant cone in $\mathbf{R}^3$. Now, since Diag is isomorphic to $\mathbf{R} \times \mathbf{Z}/2\mathbf{Z}$, it is easy to let Diag act non-trivially on $\mathbf{R}^+_*$ and the new suspension will provide a new action of $SL(2, \mathbf{R})$. If the new action of Diag extends to $\mathbf{R}^+$ and is sufficiently flat at 0, this action of $SL(2, \mathbf{R})$ can be equivariantly glued to the invariant cone and provides non-linearizable smooth actions of $SL(2, \mathbf{R})$ on $(\mathbf{R}^3, 0)$.

9. A C^∞-ACTION OF $SL(3, \mathbf{R})$ WHICH IS NOT LINEARIZABLE

We start with the adjoint action of $SL(3, \mathbf{R})$ on its Lie algebra $\mathfrak{sl}(3, \mathbf{R}) \cong \mathbf{R}^8$. Denote by Diag the subgroup of $SL(3, \mathbf{R})$ of diagonal matrices. This group is isomorphic to $\mathbf{R}^2 \times (\mathbf{Z}/2\mathbf{Z})^2$. Let *diag* $\subset \mathfrak{sl}(3, \mathbf{R})$ denote the 2-dimensional subalgebra consisting of diagonal matrices. The Weyl group, which is in this case the symmetric group on 3 letters, acts linearly on *diag* by permutation of the axis. The orbit of any point in *diag* under the adjoint action is a properly embedded submanifold of $\mathfrak{sl}(3, \mathbf{R})$ which intersects *diag* on some orbit of the Weyl group. Let C be a Weyl chamber in *diag*, for example the region consisting of diagonal matrices $(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 \leq \lambda_2 \leq \lambda_3$. This is a fundamental domain for the action of the Weyl group.

Choose a closed disc D contained in the *interior* of the Weyl chamber C. The saturation $\mathrm{Sat}(D)$ of D under the adjoint action of $SL(3,\mathbf{R})$ is a properly embedded submanifold with boundary, which fibres over D.

LEMMA 9.1. *The action obtained by suspension of the trivial action of* Diag *on the disc D is the adjoint action of $SL(3,\mathbf{R})$ on the saturation of the disc.*

Proof. This is clear since the stabilizer of any point in D under the adjoint action is precisely Diag. □

LEMMA 9.2. *There exists a C^∞-action of $\mathbf{R}^2$ on the plane with support inside the unit disc.*

Proof. Let $\rho\colon \mathbf{R}^+ \to [0,1[$ be a C^∞-diffeomorphism which is equal to the identity in a neighbourhood of 0. This defines an embedding of $\mathbf{R}^2$ in the unit disc in $\mathbf{R}^2$ sending the point of polar coordinates (r,θ) to $\big(\rho(r),\theta\big)$. Now define an action of $\mathbf{R}^2$ on $\mathbf{R}^2$ in the following way. Inside the unit disc, this action is conjugated by the previous embedding to the canonical action of $\mathbf{R}^2$ on itself by translations. Outside, the action of $\mathbf{R}^2$ is trivial. It is a simple exercise to check that with a suitable choice of ρ, one can guarantee that this action is C^∞. □

Consider such a C^∞-action of $\mathbf{R}^2$ on D whose support lies in the interior of D. This defines an action of Diag $\simeq \mathbf{R}^2 \times (\mathbf{Z}/2\mathbf{Z})^2$ on D for which $(\mathbf{Z}/2\mathbf{Z})^2$ acts trivially. By suspension, we get a C^∞-action of $SL(3,\mathbf{R})$ on some 8-manifold with boundary, which fibres over $SL(3,\mathbf{R})/\,\mathrm{Diag}$ with a closed disc as a fibre. This manifold is therefore diffeomorphic to the saturation $\mathrm{Sat}(D)$ of D under the adjoint action of $SL(3,\mathbf{R})$. The idea is to replace the adjoint action by this new action inside this manifold. However, since $\mathrm{Sat}(D)$ does not accumulate at the origin, this new action has not been modified near the origin and is therefore still locally linear. We shall therefore perform this modification on a sequence of discs in C accumulating to the origin and it will be easy to see that the action obtained in this way is not linearizable.

In order to realize this construction, we employ a family of C^∞-actions of $\mathbf{R}^2$ with support in the interior of the disc D, which depend continuously on a parameter $\epsilon \in [0,1]$ in the C^∞-topology and which is trivial when $\epsilon = 0$. This is easy to construct: just multiply the fundamental vector fields of some action of $\mathbf{R}^2$ by ϵ. We can also consider this family of actions as an action of $\mathbf{R}^2$ on $D \times [0,1]$ which is trivial on $D \times \{0\}$. By suspension, we get an action of $SL(3,\mathbf{R})$ on a 9-manifold, fibring over $SL(3,\mathbf{R})/\operatorname{Diag}$ with compact fibres $D \times [0,1]$. It is therefore diffeomorphic to $\operatorname{Sat}(D) \times [0,1]$. Therefore, we can project everything to $\operatorname{Sat}(D)$ in such a way that we get a continuous family of actions Φ_ϵ of $SL(3,\mathbf{R})$ on $\operatorname{Sat}(D)$ having the following properties. In some neighbourhood of the boundary of $\operatorname{Sat}(D)$, all these actions coincide with the adjoint action and Φ_0 is the adjoint action.

We can restate this as follows. Choose a basis of the Lie algebra $\mathfrak{sl}(3,\mathbf{R})$ and consider the associated linear vector fields $X_1,\dots,X_8$ on $\mathfrak{sl}(3,\mathbf{R}) \cong \mathbf{R}^8$ which are the corresponding infinitesimal generators of the adjoint action. Then we have C^∞ families of vector fields $X_1^\epsilon,\dots,X_8^\epsilon$ on $\operatorname{Sat}(D)$ such that $X_1^0 = X_1,\dots,X_8^0 = X_8$ and such that they satisfy the same bracket relations for all ϵ; that is, they generate an action of $\mathfrak{sl}(3,\mathbf{R})$. Denote by R_i^ϵ the difference $X_i^\epsilon - X_i$ (for $i = 1,\dots,8$). Extending by 0 outside $\operatorname{Sat}(D)$, one gets C^∞ vector fields in $\mathbf{R}^8$.

Now consider some contracting homothety Λ of $\mathbf{R}^8$ which is such that all images of the disc D under the iterates of Λ are pairwise disjoint. Note that Λ preserves each X_i since these vector fields are linear. If a sequence $(\epsilon_l)_{l\geq 0} \in [0,1]$ converges to 0 sufficiently quickly when l goes to infinity then the infinite sums

$$\bar{R}_i = \sum_{l\geq 0} \Lambda_*^l (R_i^{\epsilon_l})$$

converge uniformly on compact sets in $\mathbf{R}^8$ and define C^∞ vector fields. Now, define the vector fields $\bar{X}_i = X_i + \bar{R}_i$. In $\Lambda^l\big(\operatorname{Sat}(D)\big)$ the vector fields $\bar{X}_i$ coincide with $\Lambda_*^l(X_i^{\epsilon_l})$ and outside these regions, they are equal to X_i. It follows that the $\bar{X}_i$ satisfy the same bracket relations as the X_i, and hence generate a C^∞-action of the Lie algebra $\mathfrak{sl}(3,\mathbf{R})$ on $\mathbf{R}^8$. These vector fields are complete and generate an action of $SL(3,\mathbf{R})$ since we know that inside $\Lambda^l\big(\operatorname{Sat}(D)\big)$ they integrate to a suspension and outside they integrate to the adjoint action.

The C^∞-action of $SL(3,\mathbf{R})$ on $\mathbf{R}^8$ thus obtained is not linearizable, since it has a countable number of open 8-dimensional orbits and this is obviously not possible for a linear representation.

10. Linearizability of $SL(n,\mathbf{Z})$-actions

The purpose of this section is to prove Theorem 1.2.

THEOREM 10.1. *There are no faithful C^1-actions of $SL(n,\mathbf{Z})$ on $(\mathbf{R}^m, 0)$ for $1 \leq m < n$.*

Proof. Suppose we have a faithful C^1-action of $SL(n,\mathbf{Z})$ on $(\mathbf{R}^m, 0)$. First note that the differential of the action defines a homomorphism $D\colon SL(n,\mathbf{Z}) \to GL(m,\mathbf{R})$. According to a special case of Margulis' super-rigidity theorem, proved in [40, Theorem 6], there is a finite index subgroup Γ in $SL(n,\mathbf{Z})$ and a continuous linear representation $\rho\colon SL(n,\mathbf{R}) \to GL(m,\mathbf{R})$ such that ρ and D agree on Γ. For $1 \leq m < n$, there is no such non-trivial representation ρ so that we deduce that the restriction of D to Γ is trivial. Again, by a special case of a theorem of Margulis, proved in [40, Theorem 7], for any finite index subgroup Γ of $SL(n,\mathbf{Z})$, there is no non-trivial homomorphism from Γ to $\mathbf{R}$. Hence by Thurston's stability theorem, we deduce that the action of Γ is trivial, contradicting the faithfulness of the action. □

EXAMPLE 10.2. We now give an example of a non-linearizable C^∞-action of $SL(3,\mathbf{Z})$ on $\mathbf{R}^8$. This example is obtained simply by restricting to $SL(3,\mathbf{Z})$ the action of $SL(3,\mathbf{R})$ on $\mathbf{R}^8$ given in Section 9. This gives an action with many *discrete* orbits because by construction we have an open region where the stabilizers of the $SL(3,\mathbf{R})$-action are trivial and $SL(3,\mathbf{Z})$ is discrete in $SL(3,\mathbf{R})$. But this is impossible for the linearized action, which is the adjoint representation. To see this, first note that if $g \in \mathfrak{sl}(3,\mathbf{R})$ is diagonal, then its orbit under $SL(3,\mathbf{R})$ is $SL(3,\mathbf{R})/\operatorname{Stab}_{SL(3,\mathbf{R})}(g)$. Now for most diagonal elements g, the stabilizer $\operatorname{Stab}_{SL(3,\mathbf{R})}(g)$ is just the set of diagonal elements in $SL(3,\mathbf{R})$, and the action of $SL(3,\mathbf{Z})$ on $SL(3,\mathbf{R})/\{\text{diagonal matrices}\}$ has a dense orbit if and only if the action of the diagonal matrices on $SL(3,\mathbf{R})/SL(3,\mathbf{Z})$ has a dense orbit. But this latter condition is true, by Moore's ergodicity theorem (see [50, Theorem 2.2.6]). It follows that for the adjoint representation there is a dense set of non-trivial diagonal elements whose orbits under $SL(3,\mathbf{Z})$ are dense in their orbits under $SL(3,\mathbf{R})$ and are therefore non-discrete.

EXAMPLE 10.3. We now give an example of a non-linearizable C^ω-action of $SL(2,\mathbf{Z})$ on $\mathbf{R}^2$. Consider the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

It is well known that $SL(2,\mathbf{Z})$ is an amalgamated product of the cyclic groups generated by S and T (see for example [36, Chapter 6]). Explicitly:

$$SL(2,\mathbf{Z}) = \langle S,T \,:\, S^4 = T^6 = \mathrm{Id}, \quad S^2 = T^3 \rangle .$$

Now let $f\colon \mathbf{R} \to \mathbf{R}$ be the map $f(y) = y + y^3$ and replace T by its conjugate $t = F^{-1}TF$, where $F(x,y) = (x, f(y))$. We claim that the group G of diffeomorphisms of $\mathbf{R}^2$ generated by S and t is isomorphic to $SL(2,\mathbf{Z})$. Indeed the differential of the action of G defines a homomorphism $\phi\colon G \to SL(2,\mathbf{Z})$ which takes S to S and t to T. To construct the inverse homomorphism from $SL(2,\mathbf{Z})$ to G, it suffices to send S to S and T to t, and then check the group relations: but t clearly has order 6 and since f is an odd function, one has $t^3 = -\mathrm{Id} = S^2$.

Now let $P = S^{-1}t$. One has $P(x,y) = \left(f^{-1}(x + f(y)), f(y)\right)$. In particular, $P(x,0) = (f^{-1}(x), 0)$ and so the x-axis is an invariant line on which P is a contraction. Hence P cannot be topologically conjugate to its linear part, which is the parabolic matrix $S^{-1}T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

We now study analytic actions of lattices and prove a linearizability result analogous to Kushnirenko's theorem. We state it for general lattices rather than for the special case of $SL(n,\mathbf{Z})$ since the proof is the same.

THEOREM 10.4. *Let Γ be any irreducible lattice in a connected semi-simple Lie group with finite center, no non-trivial compact factor group and of rank bigger than 1. Every C^ω-action of Γ on $(\mathbf{R}^m, 0)$ is linearizable.*

We begin with several lemmas. We fix a lattice Γ as in the theorem and a real analytic action Φ of Γ on $(\mathbf{R}^m, 0)$.

LEMMA 10.5. *The action of Γ is formally linearizable.*

Proof. Margulis has shown that the first cohomology of Γ with values in any finite dimensional linear representation vanishes [27, Chap. IX, Theorem 6.15]. Hence the proof of Theorem 2.8 applies. □

LEMMA 10.6. *Let D be any representation of Γ in $GL(m,\mathbf{C})$. Then the traces of all the matrices in the image of D are algebraic numbers.*

Proof. This is also a well known corollary of the vanishing of first cohomology groups. One first remarks that the homomorphism D is rigid;

that is, any other homomorphism close to D on a finite system of generators is conjugate to D. This again uses the vanishing of $H^1(\Gamma, \mathfrak{gl}(m, \mathbf{C}))$ (see [27, ibid.]). Then denote by $\mathfrak{k}$ the field generated by the traces of all matrices in $D(\Gamma)$. This is a finitely generated extension of the rationals and one has to show that it is an algebraic extension. But if this was not the case, one could deform the embedding $\mathfrak{k} \subset \mathbf{C}$ by using some non-trivial Galois automorphism of $\mathbf{C}$. Applying this automorphism to all elements of $D(\Gamma)$, this would construct a non-trivial deformation of D, which is impossible. □

LEMMA 10.7. For every γ in Γ such that $D(\gamma)$ is semi-simple, the diffeomorphism $\Phi(\gamma)$ is analytically linearizable.

Proof. We recall Brjuno's linearization theorem (see [7, Chapter 11, Theorem 10] or [28, théorème 3]). Let f be an analytic diffeomorphism of $(\mathbf{R}^m, 0)$. Suppose that f is formally linearizable and that the linear part of f is a semi-simple matrix whose eigenvalues are $(\lambda_1, \dots, \lambda_m)$. If these eigenvalues satisfy some diophantine condition (Ω) described below, then f is analytically linearizable. For any positive integer k, denote by ω_k the infimum of the modulus of non-zero numbers of the form $\lambda_1^{q_1} \cdots \lambda_m^{q_m} - 1$ where the q_i are integers such that $q_i \geq -1$, at most one of the q_i equals -1, and $\sum_i q_i \leq 2^{k+1}$. Then the condition (Ω) asserts that the series $\sum_{k \geq 1} 2^{-k} \ln \omega_k^{-1}$ converges.

According to Lemma 10.5, the diffeomorphism $\Phi(\gamma)$ is formally linearizable. According to Lemma 10.6, all eigenvalues $(\lambda_1, \dots, \lambda_m)$ of the differential $D(\gamma)$ of $\Phi(\gamma)$ at the origin are algebraic numbers. An important theorem of Baker shows that there is a constant $C > 0$ such that for all integers k, we have $\omega_k \geq \exp(-Ck)$ [3, Theorem 3.1]. It follows that the condition (Ω) is satisfied and one can apply Brjuno's theorem. □

REMARK 10.8. In most cases, the spectrum of $D(\gamma)$ contains many resonances. Not only the determinant of $D(\gamma)$ is one since there is no non-trivial homomorphisms from Γ to $\mathbf{R}$ but there are extra resonances coming from the structure of linear representations. Suppose for example that $\Gamma = SL(n, \mathbf{Z})$ and that $\Phi = D$ is the restriction to Γ of a linear representation of $SL(n, \mathbf{R})$ in $GL(m, \mathbf{R})$. Then the many integral linear relations between the weights of this representation provide corresponding multiplicative relations between the eigenvalues of the matrix $D(\gamma)$. Hence, in order to prove the previous lemma, the classical linearization theorem of Siegel is not sufficient ([28]): one has to use the more powerful theorem of Brjuno which allows resonances but it was indeed necessary to first prove the formal linearizability.

Of course, our problem now is that the diffeomorphisms which linearize the $\Phi(\gamma)$ might depend on γ. The difficulty comes again from the resonances since these imply that the centralizers of $D(\gamma)$ are big inside the group of analytic diffeomorphisms.

Denote by $\mathrm{Diff}(\mathbf{R}^m, 0)$ the group of germs of real analytic diffeomorphisms of $\mathbf{R}^m$ at 0 and by $\widehat{\mathrm{Diff}}(\mathbf{R}^m, 0)$ the group of formal diffeomorphisms. We can consider Φ as a homomorphism from Γ to $\mathrm{Diff}(\mathbf{R}^m, 0) \subset \widehat{\mathrm{Diff}}(\mathbf{R}^m, 0)$. The linear part D of Φ is a homomorphism from Γ to $GL(m, \mathbf{R})$.

We can assume that $D(\Gamma)$ is infinite. Indeed, if $D(\Gamma)$ is finite, the kernel of D acts trivially by Thurston's theorem so that the action Φ factors through a finite group and is therefore linearizable.

By Lemma 10.5, there is an element $\widehat{f}$ in $\widehat{\mathrm{Diff}}(\mathbf{R}^m, 0)$ which conjugates Φ and D. Let $H \subset GL(m, \mathbf{R})$ be the Zariski closure of $D(\Gamma)$. According to [27, ibid.], H is a semi-simple group. Let $\phi\colon H \to \widehat{\mathrm{Diff}}(\mathbf{R}^m, 0)$ be defined by $\phi(h) = \widehat{f}h\widehat{f}^{-1}$ so that for $\gamma \in \Gamma$, we have $\Phi(\gamma) = \phi\big(D(\gamma)\big)$. If we could show that $\phi(H) \subset \mathrm{Diff}(\mathbf{R}^m, 0)$ then we could apply Kushnirenko's theorem and there would exist an element f of $\mathrm{Diff}(\mathbf{R}^m, 0)$ such that $f\phi(H)f^{-1}$ is contained in $GL(m, \mathbf{R})$. Since $f\Phi(\gamma)f^{-1} = f\phi\big(D(\gamma)\big)f^{-1}$ the convergent diffeomorphism f would linearize $\Phi(\Gamma)$ as required.

Therefore, we denote by $H_0 \subset H$ the inverse image of $\mathrm{Diff}(\mathbf{R}^m, 0)$ by ϕ and we shall show that $H_0 = H$. Observe first that obviously $D(\Gamma)$ is contained in H_0 since $\phi\big(D(\gamma)\big) = \Phi(\gamma)$ is convergent by hypothesis.

For each γ in Γ, denote by $\langle D(\gamma)\rangle$ the Zariski closure of the group generated by $D(\gamma)$ in $GL(m, \mathbf{R})$. We claim that $\langle D(\gamma)\rangle$ is contained in H_0 if $D(\gamma)$ is semi-simple.

Indeed, by Lemma 10.7, we know that there is a convergent diffeomorphism f_γ such that $f_\gamma\Phi(\gamma)f_\gamma^{-1} = D(\gamma)$. The algebraic group consisting of those elements g of $GL(m, \mathbf{R})$ such that $f_\gamma\phi(g)f_\gamma^{-1} = g$ contains $D(\gamma)$, hence $\langle D(\gamma)\rangle$. It follows that every element of $\langle D(\gamma)\rangle$ has an image under ϕ which is conjugate by f_γ to a linear map so that in particular $\phi(\langle D(\gamma)\rangle)$ consists of convergent diffeomorphisms and $\langle D(\gamma)\rangle$ is indeed contained in H_0 as we claimed.

Observe that by Remark 2.1 we can replace Γ by a subgroup of finite index. In particular, using Selberg's lemma, we can assume that $D(\Gamma)$ is torsion free and, more precisely, that if some power of some $D(\gamma)$ lies in a normal subgroup of H then $D(\gamma)$ is in this subgroup (note that there are finitely many such normal subgroups).

Since $D(\gamma)$ has infinite order (if γ is non-trivial), $\langle D(\gamma)\rangle$ has positive dimension so that it contains a non-trivial one-parameter group. Hence every non trivial semi-simple element in $D(\Gamma)$ yields a one-parameter group contained in H_0. We now show that these one-parameter subgroups generate the connected component of the identity in H. Observe the following elementary fact: if a family of vectors spans the Lie algebra of a Lie group, then the one-parameter groups generated by these vectors generate the connected component of the identity. Therefore, we consider the linear span $\mathfrak{C}$ in the Lie algebra $\mathfrak{H}$ of H of the Lie algebras of all the subgroups $\langle D(\gamma)\rangle$ for γ semi-simple. It is enough to show that $\mathfrak{C} = \mathfrak{H}$. Note that $\mathfrak{C}$ is certainly non-trivial since semi-simple elements are Zariski dense in H. Note also that $\mathfrak{C}$ is invariant under the adjoint action of $D(\Gamma)$, hence under the adjoint action of H since $D(\Gamma)$ is Zariski dense in H. It follows that $\mathfrak{C}$ coincides with the product of some of the simple factors of $\mathfrak{H}$. The only possibility is that $\mathfrak{C} = \mathfrak{H}$ since otherwise, all the semi-simple $D(\gamma)$ would have some power contained in the same product of some but not all of the simple factors of H (note that the algebraic Abelian group $\langle D(\gamma)\rangle$ has a finite number of connected components). This implies that all semi-simple elements of $D(\Gamma)$ are contained in some non trivial normal subgroup of H. This is not possible by the following argument. In the algebraic group H, there is a non-empty open Zariski set consisting of semi-simple elements which are not contained in any non-trivial normal subgroup of H. Since $D(\Gamma)$ is Zariski dense in H, it intersects non-trivially this open set.

It follows that H_0 contains the connected component of the identity of H. Therefore H_0 is a semi-simple Lie group of finite index in H. By Kushnirenko's theorem, we can analytically linearize $\phi(H)$ (one also uses Remark 2.1) and in particular $\Phi(\Gamma)$.

Theorem 10.4 is proved.

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