5. The adjoint representation of SL(2,R)

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It follows that after conjugation by F, the $SL(n, \mathbf{R})$-action commutes with dilations. Indeed, consider the conjugated $SL(n, \mathbf{R})$-action. If $f \in SL(n, \mathbf{R})$, $x \in \mathbf{R}^n$ and $\lambda > 0$, then choose $a, b \in SO(n)$ such that $ax \in \Delta^+$ and $bf(\lambda x) \in \Delta^+$. Provided x is sufficiently close to 0, ax and $bf(\lambda x)$ will lie in the domain of f. Then $bfa^{-1} \in H$ and so

$$\begin{aligned} f(\lambda x) &= b^{-1} bfa^{-1} a(\lambda x) = b^{-1}(bfg^{-1})\,\lambda a(x) \\ &= b^{-1}\lambda(bfa^{-1})\,a(x) = \lambda b^{-1}(bfa^{-1})\,a(x) \\ &= \lambda f(x)\,. \end{aligned}$$

The proof of the theorem is then completed by the following well known result (cf. [17, Lemma 2.1.4]). □

LEMMA 4.3. *Every C^1 map commuting with dilations is linear.*

Proof. Suppose that f is a C^1-diffeomorphism of $\mathbf{R}^n$ which commutes with dilations. By comparing the differential of $\lambda . f$ and $f \circ \lambda$ at x we have $\lambda df|_x = \lambda df|_{\lambda x}$, for each $\lambda > 0$ and every $x \in \mathbf{R}^n$. Hence $df|_x = df|_{\lambda x}$ and so df is constant on the radial lines. Thus $df|_x = df|_0$ for all x and so f is linear. □

5. THE ADJOINT REPRESENTATION OF $SL(2, \mathbf{R})$

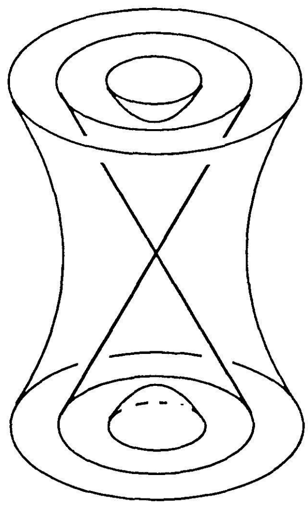
Let us recall some facts concerning the linear representations of $SL(2, \mathbf{R})$. Let $P_l(\mathbf{R}^2)$ denote the space of real valued homogeneous polynomials, of two variables, of degree l. As a vector space, $P_l(\mathbf{R}^2) \cong \mathbf{R}^{l+1}$, and the action of $SL(2, \mathbf{R})$ on $\mathbf{R}^2$ defines a linear action on $P_l(\mathbf{R}^2)$: up to isomorphism, this is the (unique) irreducible representation of $SL(2, \mathbf{R})$ in dimension $l+1$. In dimension 3, there is another useful realization of the polynomial representation, called the *adjoint representation*. Notice that the group $SL(2, \mathbf{R})$ acts by the adjoint representation on its Lie algebra $\mathfrak{sl}(2, \mathbf{R})$. Of course, $\mathfrak{sl}(2, \mathbf{R})$ is the space of 2×2 real traceless matrices; so as a vector space, $\mathfrak{sl}(2, \mathbf{R}) \cong \mathbf{R}^3$. The adjoint representation $Ad\colon SL(2, \mathbf{R}) \to GL(3, \mathbf{R})$, defined by

$$Ad(g)\colon h \mapsto ghg^{-1}, \quad \forall g \in SL(2, \mathbf{R}),\ h \in \mathfrak{sl}(2, \mathbf{R})\,,$$

is an irreducible linear representation. In fact, an explicit equivariant isomorphism $\psi\colon \mathfrak{sl}(2, \mathbf{R}) \to P_2(\mathbf{R}^2)$ is obtained by taking $\psi(h)$, as a function of variables x and y, to be the area of the parallelogram spanned by (x, y) and $h(x, y)$. That is,

$$\psi\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = by^2 + 2axy - cx^2 .$$

Recall that the Cartan-Killing form K of a semi-simple Lie algebra is a non-degenerate quadratic form which is invariant under the adjoint representation of the associated Lie group. For $\mathfrak{sl}(2,\mathbf{R})$, one has $K = -8\det$. (The Cartan-Killing form is unique up to constant factor: the factor here of -8 corresponds to the usual convention $K = tr\, Ad^2$.) Notice that in particular, K has signature $(-,+,+)$ and hence determines a Minkowski metric on $\mathfrak{sl}(2,\mathbf{R})$. The time-like elements $h \in \mathfrak{sl}(2,\mathbf{R})$ (those with $\det h > 0$) are called *elliptic* elements. The space-like, resp. light-like, elements (that is, those with $\det h < 0$, resp. $\det h = 0$) are said to be *hyperbolic*, resp. *parabolic*. Notice that under ψ, the elliptic elements correspond to quadratics which are irreducible over $\mathbf{R}$, the hyperbolic elements correspond to products of distinct linear factors, and the parabolic elements correspond to (± 1 times) the squares of linear factors. Moreover, this equips Minkowski space with a "temporal" orientation: the parabolic elements which are squares of linear factors belong to the *future*.



Orbits of the adjoint representation

We denote the exponential map by $\exp\colon \mathfrak{sl}(2,\mathbf{R}) \to SL(2,\mathbf{R})$. It is common to say that $g = \exp h$ is parabolic, resp. elliptic, resp. hyperbolic, according to the type of h. The parabolic elements $g \in SL(2,\mathbf{R})$ are those with $tr^2(g) = 4$, the elliptic elements have $tr^2(g) < 4$, and the hyperbolic elements have $tr^2(g) > 4$. Notice also that the universal cover $\widetilde{SL}(2,\mathbf{R})$ of $SL(2,\mathbf{R})$ is also a Lie group. Let us denote the corresponding exponential map by $\widetilde{\exp}\colon \mathfrak{sl}(2,\mathbf{R}) \to \widetilde{SL}(2,\mathbf{R})$. The kernel of the natural quotient map $\widetilde{SL}(2,\mathbf{R}) \to SL(2,\mathbf{R})$ is precisely the image under $\widetilde{\exp}$ of the elliptic elements

$$\left\{ 2\pi n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \; : \; n \in \mathbf{Z} \right\} .$$

Now consider the orbits of the points $h \in \mathfrak{sl}(2, \mathbf{R})$ under the adjoint representation of $SL(2, \mathbf{R})$. Notice that since this action leaves K invariant, the action preserves the spheres $K =$ constant, in Minkowski space $\mathbf{R}^{1,2}$. (Of course, these Minkowski "spheres" are hyperboloids of revolution in $\mathbf{R}^3$. See figure.) So the orbits of the adjoint representation lie in these Minkowski spheres. In fact, it is easy to see that the orbits are precisely the connected components of these Minkowski spheres. (This is essentially the Jordan canonical form theorem in dimension 2.) In the case of non-zero parabolic elements, this means that the orbits are precisely the connected components of the light-cone minus the origin. Typical stabilizers of the adjoint representation are:

$$\text{hyperbolic case:} \quad \mathrm{Stab}_{SL(2,\mathbf{R})}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \left\{ \pm \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbf{R} \right\}$$

$$\text{parabolic case:} \quad \mathrm{Stab}_{SL(2,\mathbf{R})}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \left\{ \pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbf{R} \right\}$$

$$\text{elliptic case:} \quad \mathrm{Stab}_{SL(2,\mathbf{R})}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} : t \in \mathbf{R} \right\}.$$

For every non-zero element $h \in \mathfrak{sl}(2, \mathbf{R})$, the stabilizer $\mathrm{Stab}_{SL(2,\mathbf{R})}(h)$ is (± 1 times) the one-parameter subgroup $\{\exp(th) : t \in \mathbf{R}\}$ generated by h. Notice that if $h \in \mathfrak{sl}(2, \mathbf{R})$ is elliptic (resp. hyperbolic or parabolic), then $\mathrm{Stab}_{SL(2,\mathbf{R})}(h)$ is a circle (resp. two lines).

6. $SL(2, \mathbf{R})$-ACTIONS ON $\mathbf{R}^2$

By Theorem 3.5, the only homogeneous space of $SL(2, \mathbf{R})$ of dimension 1 on which $SL(2, \mathbf{R})$ acts faithfully is the circle S^1 equipped with the projective action. We now examine the homogeneous spaces of $SL(2, \mathbf{R})$ of dimension 2.

LEMMA 6.1. *Every faithful transitive action of $SL(2, \mathbf{R})$ on a noncompact surface is conjugate to one of the following two actions:*

(*a*) *the canonical action on* $SL(2, \mathbf{R}) / \left\{ \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} : t > 0 \right\} \cong \mathbf{R}^2 \setminus \{0\}$,

(*b*) *the canonical action on* $SL(2, \mathbf{R}) / \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbf{R} \right\} \cong \mathbf{R}^2 \setminus \{0\}$.

Proof. Of course, the homogeneous spaces of $SL(2, \mathbf{R})$ of dimension 2 are determined by the closed subgroups of $SL(2, \mathbf{R})$ of dimension 1. The