6. SL(2,R) -actions on R^2

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Now consider the orbits of the points $h \in \mathfrak{sl}(2, \mathbf{R})$ under the adjoint representation of $SL(2, \mathbf{R})$. Notice that since this action leaves K invariant, the action preserves the spheres $K =$ constant, in Minkowski space $\mathbf{R}^{1,2}$. (Of course, these Minkowski "spheres" are hyperboloids of revolution in $\mathbf{R}^3$. See figure.) So the orbits of the adjoint representation lie in these Minkowski spheres. In fact, it is easy to see that the orbits are precisely the connected components of these Minkowski spheres. (This is essentially the Jordan canonical form theorem in dimension 2.) In the case of non-zero parabolic elements, this means that the orbits are precisely the connected components of the light-cone minus the origin. Typical stabilizers of the adjoint representation are:

$$\text{hyperbolic case:} \quad \mathrm{Stab}_{SL(2,\mathbf{R})}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \left\{ \pm \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbf{R} \right\}$$

$$\text{parabolic case:} \quad \mathrm{Stab}_{SL(2,\mathbf{R})}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \left\{ \pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbf{R} \right\}$$

$$\text{elliptic case:} \quad \mathrm{Stab}_{SL(2,\mathbf{R})}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} : t \in \mathbf{R} \right\}.$$

For every non-zero element $h \in \mathfrak{sl}(2, \mathbf{R})$, the stabilizer $\mathrm{Stab}_{SL(2,\mathbf{R})}(h)$ is (± 1 times) the one-parameter subgroup $\{\exp(th) : t \in \mathbf{R}\}$ generated by h. Notice that if $h \in \mathfrak{sl}(2, \mathbf{R})$ is elliptic (resp. hyperbolic or parabolic), then $\mathrm{Stab}_{SL(2,\mathbf{R})}(h)$ is a circle (resp. two lines).

6. $SL(2, \mathbf{R})$-ACTIONS ON $\mathbf{R}^2$

By Theorem 3.5, the only homogeneous space of $SL(2, \mathbf{R})$ of dimension 1 on which $SL(2, \mathbf{R})$ acts faithfully is the circle S^1 equipped with the projective action. We now examine the homogeneous spaces of $SL(2, \mathbf{R})$ of dimension 2.

LEMMA 6.1. *Every faithful transitive action of* $SL(2, \mathbf{R})$ *on a noncompact surface is conjugate to one of the following two actions:*

(*a*) *the canonical action on* $SL(2, \mathbf{R}) / \left\{ \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} : t > 0 \right\} \cong \mathbf{R}^2 \setminus \{0\}$,

(*b*) *the canonical action on* $SL(2, \mathbf{R}) / \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbf{R} \right\} \cong \mathbf{R}^2 \setminus \{0\}$.

Proof. Of course, the homogeneous spaces of $SL(2, \mathbf{R})$ of dimension 2 are determined by the closed subgroups of $SL(2, \mathbf{R})$ of dimension 1. The connected component of a closed subgroup of $SL(2,\mathbf{R})$ of dimension 1 is a one-parameter subgroup: so it is either hyperbolic parabolic, or elliptic. This gives the following three homogeneous spaces:

(a) $$SL(2,\mathbf{R})/\left\{\begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} : t>0\right\} \cong \mathbf{R}^2\backslash\{0\},$$

(b) $$SL(2,\mathbf{R})/\left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t\in\mathbf{R}\right\} \cong \mathbf{R}^2\backslash\{0\},$$

(c) $$SL(2,\mathbf{R})/\left\{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta\in\mathbf{R}\right\} \cong \mathbf{R}^2.$$

Up to a twofold covering, these actions are just the restrictions of the adjoint representation to the orbits seen in the previous section. Notice however that in the elliptic case the element $-\mathrm{Id}$ acts trivially, and so the action is not faithful. So this leaves the two required actions.

It remains to show that the homogeneous spaces of the form $SL(2,\mathbf{R})/H$, where H is *not* connected, do not give us any new faithful actions. But it is easy to see that in the hyperbolic case, there are only two possibilities, corresponding to H having 2 or 4 connected components, and $-\mathrm{Id}$ acts trivially in each case. In the parabolic case, the situation is similar to that of Part (c) of Theorem 3.5: either $-\mathrm{Id}$ acts trivially, or the homogeneous space is compact. $\square$

We now classify the continuous $SL(2,\mathbf{R})$-actions on $\mathbf{R}^2$. As in the higher dimensional case, we do this by giving a recipe for constructing examples, and then prove that this gives a complete list.

First, consider the oriented annulus $A=\{(r,\theta) : 1/2<r<2\}$, expressed in polar coordinates. Note that the above lemma furnishes us with three faithful transitive actions of $SL(2,\mathbf{R})$ on A. By conjugation by the map $\psi\colon \mathbf{R}^2\backslash\{0\}\to A$ defined by $\psi(r,\theta)=\left(\frac{1+2r}{2+r},\theta\right)$, the action (b) on $\mathbf{R}^2\backslash\{0\}$ gives us an action on A which we denote $\mathcal{P}^+$. By conjugating this by the inversion $(r,\theta)\mapsto(1/r,\theta)$, we obtain another action, which we denote $\mathcal{P}^-$. In the hyperbolic case (a), the above lemma gives us another action, which we denote $\mathcal{H}$, but it is easy to see that in this case, inversion gives us an isomorphic action.

Now choose a closed set $S\subset\mathbf{R}_*^+$ and choose a continuous function $T\colon\mathbf{R}_*^+\backslash S\to\{-1,0,1\}$. Then one obtains an $SL(2,\mathbf{R})$-action $\Phi_{S,T}$ on $(\mathbf{R}^2,0)$ as follows: taking $\mathbf{R}_*^+$ to be the radial coordinate, for each $s\in S$ one takes the circle of radius s to be a one-dimensional orbit, equipped with the canonical projective action, and for each connected component C of $\mathbf{R}^+_* \backslash S$, one takes an action $\mathcal{P}^+, \mathcal{P}^-$ or $\mathcal{H}$ according to whether $T(C)$ is $1, -1$ or 0 respectively. It is easy to see that the actions on the two-dimensional orbits agree on their boundaries with the action on the one-dimensional orbit, so one does indeed obtain a continuous action.

THEOREM 6.2. *Every faithful C^0-action of $SL(2, \mathbf{R})$ on $(\mathbf{R}^2, 0)$ is conjugate to one of the above actions $\Phi_{S,T}$.*

Proof. First we linearize the $SO(2)$-action, using Proposition 3.8. This shows that the origin is the only zero-dimensional orbit, and that the one-dimensional orbits are circles centred at the origin. Moreover, from above, the restricted $SL(2, \mathbf{R})$-action on the one-dimensional orbits is the canonical projective action, and the actions on the two-dimensional orbits are each individually conjugate to either $\mathcal{P}^+, \mathcal{P}^-$ or $\mathcal{H}$. It remains to see that the open orbits can be glued to their boundaries in a unique manner.

Notice that if x lies in a one-dimensional orbit Ω, then $\mathrm{Stab}_{SL(2,\mathbf{R})}(x)$ contains a unique one-parameter parabolic subgroup G_x of $SL(2, \mathbf{R})$, and conversely, each one-parameter parabolic subgroup G_x fixes a unique pair of points $\pm x \in \Omega$. Inside the orbits of $\mathcal{P}^+$ and $\mathcal{P}^-$, the fixed point sets of the subgroups G_x are radial lines passing from one boundary component of the annulus to the other component. It follows that each end can be glued to a circle in precisely two ways which respect the action of the one-parameter parabolic subgroups. In fact, since $-\mathrm{Id}$ commutes with the $SL(2, \mathbf{R})$-action, the resulting actions are isomorphic.

Similarly, one treats the hyperbolic two-dimensional orbit of $\mathcal{H}$ by considering the fixed points sets of the one-parameter hyperbolic subgroups of $SL(2, \mathbf{R})$. If Ω is a one-dimensional orbit, then each one-parameter hyperbolic subgroup fixes four points in Ω. Conversely, each point $x \in \Omega$ is fixed by a family F_x of one-parameter hyperbolic subgroups. For the action $\mathcal{H}$, the one-parameter hyperbolic subgroups are the stabilizers of the points, and each one-parameter hyperbolic subgroup has precisely four fixed points. For each $x \in \Omega$, the fixed points of the elements of F_x define four curves which pass from one boundary component of the annulus to the other. It is not difficult to see that a unique $SL(2, \mathbf{R})$-action results by gluing each end of the annulus to a circle in such a way as to have continuity of these fixed point sets. □

We now complete the proof of Theorem 1.1.

THEOREM 6.3. *For all $k = 1, \dots, \infty$, every C^k-action of $SL(2, \mathbf{R})$ on $(\mathbf{R}^2, 0)$ is C^k-linearizable.*

Proof. The proof is essentially the same as that of Theorem 4.2, except that we require a replacement for Lemma 3.9. Of course, it is not true that two points of $\mathbf{R}^2$ lie in the same radial line if and only if they have the same stabilizer under the $SO(2)$-action. The idea is to instead use the stable manifolds of the hyperbolic elements of $SL(2, \mathbf{R})$.

Let $\Phi \colon SL(2, \mathbf{R}) \to \mathrm{Diff}(\mathbf{R}^2, 0)$ be our given C^1-action. First note that as in the proof of Theorem 4.2, we may assume that locally the $SO(2)$-action is the canonical linear one and that the differential of Φ at the origin is the identity. Now let

$$h^t = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}$$

and consider the hyperbolic flow $\phi^t = \Phi(h^t)$ on $(\mathbf{R}^2, 0)$. By the stable manifold theorem (see [17, Theorem 6.2.8 and Theorem 17.4.3]), the stable manifold S_0 of ϕ^t is locally the graph of a C^1-function from $(\mathbf{R}, 0)$ to $(\mathbf{R}, 0)$. It follows that there is a local C^1-diffeomorphism of $(\mathbf{R}^2, 0)$ which commutes with the $SO(2)$-action and which takes S_0 to the x-axis. Conjugating Φ by this diffeomorphism, we may assume that locally S_0 is the x-axis. Then by using Theorem 2.5 we may linearize the action of ϕ^t on S_0, with some local C^k-diffeomorphism f of the x-axis and then extend the conjugation to $(\mathbf{R}^2, 0)$, using Equation (2) of Section 4. The upshot of this is that we may assume that, at least locally, the $SO(2)$-action is the canonical one, and the action of the subgroup $H = \{h^t : t \in \mathbf{R}\} \subset SL(2, \mathbf{R})$ is linear on the x-axis.

We will show that the $SL(2, \mathbf{R})$-action now preserves the radial lines. Let $R_\theta \in SO(2)$ denote the rotation through angle θ and let $f^t_\theta = R_\theta h^t R_\theta^{-1}$. Then clearly the stable manifold S_θ of $\Phi(f^t_\theta)$ is the radial line at angle θ. Now let $g \in SL(2, \mathbf{R})$ and consider $\Sigma = \Phi(g)(S_\theta)$. We want to show that Σ is a radial line. Clearly Σ is the stable manifold of the hyperbolic flow $\Phi(g f^t_\theta g^{-1})$. Let σ denote the angle of the stable line of the hyperbolic one-parameter group of matrices $g f^t_\theta g^{-1}$. Then $R_\sigma^{-1}\Sigma$ is the stable manifold Σ_A of the hyperbolic flow $\Phi(A^t)$, where $A^t = R_\sigma^{-1} g f^t_\theta g^{-1} R_\sigma$. Now the stable line of the hyperbolic flow A^t is the x-axis; that is A^t is a one-parameter subgroup of the form

$$A^t = \exp(t \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}) = \begin{pmatrix} e^{-t} & -b \sinh t \\ 0 & e^t \end{pmatrix}$$

for some $b \in \mathbf{R}$. We are required to show that Σ_A is the x-axis. First notice that *restricted to the x-axis*, one has

$$\begin{aligned}\Phi(A^t) &= \Phi\left(\begin{pmatrix}1 & b(e^{-2t}-1)/2\\ 0 & 1\end{pmatrix}\begin{pmatrix}e^{-t} & 0\\ 0 & e^t\end{pmatrix}\right)\\ &= \Phi\left(\begin{pmatrix}1 & b(e^{-2t}-1)/2\\ 0 & 1\end{pmatrix}\right)\begin{pmatrix}e^{-t} & 0\\ 0 & e^t\end{pmatrix}\end{aligned}$$

since H acts linearly on the x-axis. Hence, since the family of maps

$$F_t = \Phi\left(\begin{pmatrix}1 & b(e^{-2t}-1)/2\\ 0 & 1\end{pmatrix}\right) \qquad t \geq 0$$

is equicontinuous in some neighbourhood of the identity, we conclude that Σ_A is the x-axis, as required.

By the above argument, we may assume that locally the $SO(2)$-action is the canonical one and the $SL(2,\mathbf{R})$-action preserves the radial lines. The proof is then completed as in the proof of Theorem 4.2. □

7. Examples of C^0-actions of $SL(2,\mathbf{R})$ on $\mathbf{R}^m$

When m is greater than n there is a plethora of examples of continuous actions of $SL(n,\mathbf{R})$ on $(\mathbf{R}^m, 0)$. In this section we give some examples in the case $n = 2$.

7.1. The symmetric product. Choose one of the continuous $SL(2,\mathbf{R})$-actions on $(\mathbf{R}^2, 0)$ from the previous section. Now consider the associated $SL(2,\mathbf{R})$-action on the symmetric product

$$\Pi_{i=1}^m \mathbf{R}^2/\Sigma_m \cong \mathbf{C}^m,$$

where Σ_m is the symmetric group on m letters. Recall that the last identification associates to an m-tuple of points $(x_1, \ldots, x_m)$ in $\mathbf{R}^2 \cong \mathbf{C}$ the coefficients of the monic polynomial of degree m in one complex variable whose roots are the x_i. As the original action fixed the origin in $\mathbf{R}^2$, so the corresponding action fixes the origin in $\mathbf{R}^{2m}$.

7.2. The adjoint action at infinity. Consider the adjoint action of $SL(2,\mathbf{R})$ on $\mathbf{R}^3$, as discussed in Section 5. Removing the origin and compactifying the other end, we obtain a C^0-action of $SL(2,\mathbf{R})$ on $\mathbf{R}^3$, which we will call the *adjoint action at infinity*. This action is certainly not topologically linearizable, since all the orbits now accumulate to the fixed point. In fact, this action is not topologically conjugate to any C^1-action. To see this, consider the hyperbolic element $h = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Using the exponential