10. LINEARIZABILITY OF SL(n,Z)-ACTIONS

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10. Linearizability of $SL(n, \mathbf{Z})$-actions

The purpose of this section is to prove Theorem 1.2.

THEOREM 10.1. *There are no faithful C^1-actions of $SL(n, \mathbf{Z})$ on $(\mathbf{R}^m, 0)$ for $1 \leq m < n$.*

Proof. Suppose we have a faithful C^1-action of $SL(n, \mathbf{Z})$ on $(\mathbf{R}^m, 0)$. First note that the differential of the action defines a homomorphism $D\colon SL(n, \mathbf{Z}) \to GL(m, \mathbf{R})$. According to a special case of Margulis' super-rigidity theorem, proved in [40, Theorem 6], there is a finite index subgroup Γ in $SL(n, \mathbf{Z})$ and a continuous linear representation $\rho\colon SL(n, \mathbf{R}) \to GL(m, \mathbf{R})$ such that ρ and D agree on Γ. For $1 \leq m < n$, there is no such non-trivial representation ρ so that we deduce that the restriction of D to Γ is trivial. Again, by a special case of a theorem of Margulis, proved in [40, Theorem 7], for any finite index subgroup Γ of $SL(n, \mathbf{Z})$, there is no non-trivial homomorphism from Γ to $\mathbf{R}$. Hence by Thurston's stability theorem, we deduce that the action of Γ is trivial, contradicting the faithfulness of the action. □

EXAMPLE 10.2. We now give an example of a non-linearizable C^∞-action of $SL(3, \mathbf{Z})$ on $\mathbf{R}^8$. This example is obtained simply by restricting to $SL(3, \mathbf{Z})$ the action of $SL(3, \mathbf{R})$ on $\mathbf{R}^8$ given in Section 9. This gives an action with many *discrete* orbits because by construction we have an open region where the stabilizers of the $SL(3, \mathbf{R})$-action are trivial and $SL(3, \mathbf{Z})$ is discrete in $SL(3, \mathbf{R})$. But this is impossible for the linearized action, which is the adjoint representation. To see this, first note that if $g \in \mathfrak{sl}(3, \mathbf{R})$ is diagonal, then its orbit under $SL(3, \mathbf{R})$ is $SL(3, \mathbf{R}) / \operatorname{Stab}_{SL(3,\mathbf{R})}(g)$. Now for most diagonal elements g, the stabilizer $\operatorname{Stab}_{SL(3,\mathbf{R})}(g)$ is just the set of diagonal elements in $SL(3, \mathbf{R})$, and the action of $SL(3, \mathbf{Z})$ on $SL(3, \mathbf{R})/\{\text{diagonal matrices}\}$ has a dense orbit if and only if the action of the diagonal matrices on $SL(3, \mathbf{R})/SL(3, \mathbf{Z})$ has a dense orbit. But this latter condition is true, by Moore's ergodicity theorem (see [50, Theorem 2.2.6]). It follows that for the adjoint representation there is a dense set of non-trivial diagonal elements whose orbits under $SL(3, \mathbf{Z})$ are dense in their orbits under $SL(3, \mathbf{R})$ and are therefore non-discrete.

EXAMPLE 10.3. We now give an example of a non-linearizable C^ω-action of $SL(2, \mathbf{Z})$ on $\mathbf{R}^2$. Consider the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

It is well known that $SL(2,\mathbf{Z})$ is an amalgamated product of the cyclic groups generated by S and T (see for example [36, Chapter 6]). Explicitly:

$$SL(2,\mathbf{Z}) = \left\langle S,T \,:\, S^4 = T^6 = \mathrm{Id}, \quad S^2 = T^3 \right\rangle .$$

Now let $f\colon \mathbf{R} \to \mathbf{R}$ be the map $f(y) = y + y^3$ and replace T by its conjugate $t = F^{-1}TF$, where $F(x,y) = \big(x, f(y)\big)$. We claim that the group G of diffeomorphisms of $\mathbf{R}^2$ generated by S and t is isomorphic to $SL(2,\mathbf{Z})$. Indeed the differential of the action of G defines a homomorphism $\phi\colon G \to SL(2,\mathbf{Z})$ which takes S to S and t to T. To construct the inverse homomorphism from $SL(2,\mathbf{Z})$ to G, it suffices to send S to S and T to t, and then check the group relations: but t clearly has order 6 and since f is an odd function, one has $t^3 = -\mathrm{Id} = S^2$.

Now let $P = S^{-1}t$. One has $P(x,y) = \big(f^{-1}\big(x + f(y)\big), f(y)\big)$. In particular, $P(x,0) = (f^{-1}(x), 0)$ and so the x-axis is an invariant line on which P is a contraction. Hence P cannot be topologically conjugate to its linear part, which is the parabolic matrix $S^{-1}T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

We now study analytic actions of lattices and prove a linearizability result analogous to Kushnirenko's theorem. We state it for general lattices rather than for the special case of $SL(n,\mathbf{Z})$ since the proof is the same.

THEOREM 10.4. *Let Γ be any irreducible lattice in a connected semi-simple Lie group with finite center, no non-trivial compact factor group and of rank bigger than 1. Every C^{ω}-action of Γ on $(\mathbf{R}^m, 0)$ is linearizable.*

We begin with several lemmas. We fix a lattice Γ as in the theorem and a real analytic action Φ of Γ on $(\mathbf{R}^m, 0)$.

LEMMA 10.5. *The action of Γ is formally linearizable.*

Proof. Margulis has shown that the first cohomology of Γ with values in any finite dimensional linear representation vanishes [27, Chap. IX, Theorem 6.15]. Hence the proof of Theorem 2.8 applies. □

LEMMA 10.6. *Let D be any representation of Γ in $GL(m,\mathbf{C})$. Then the traces of all the matrices in the image of D are algebraic numbers.*

Proof. This is also a well known corollary of the vanishing of first cohomology groups. One first remarks that the homomorphism D is rigid;

that is, any other homomorphism close to D on a finite system of generators is conjugate to D. This again uses the vanishing of $H^1(\Gamma, \mathfrak{gl}(m, \mathbf{C}))$ (see [27, ibid.]). Then denote by $\mathfrak{k}$ the field generated by the traces of all matrices in $D(\Gamma)$. This is a finitely generated extension of the rationals and one has to show that it is an algebraic extension. But if this was not the case, one could deform the embedding $\mathfrak{k} \subset \mathbf{C}$ by using some non-trivial Galois automorphism of $\mathbf{C}$. Applying this automorphism to all elements of $D(\Gamma)$, this would construct a non-trivial deformation of D, which is impossible. □

LEMMA 10.7. For every γ in Γ such that $D(\gamma)$ is semi-simple, the diffeomorphism $\Phi(\gamma)$ is analytically linearizable.

Proof. We recall Brjuno's linearization theorem (see [7, Chapter 11, Theorem 10] or [28, théorème 3]). Let f be an analytic diffeomorphism of $(\mathbf{R}^m, 0)$. Suppose that f is formally linearizable and that the linear part of f is a semi-simple matrix whose eigenvalues are $(\lambda_1, \ldots, \lambda_m)$. If these eigenvalues satisfy some diophantine condition (Ω) described below, then f is analytically linearizable. For any positive integer k, denote by ω_k the infimum of the modulus of non-zero numbers of the form $\lambda_1^{q_1} \cdots \lambda_m^{q_m} - 1$ where the q_i are integers such that $q_i \geq -1$, at most one of the q_i equals -1, and $\sum_i q_i \leq 2^{k+1}$. Then the condition (Ω) asserts that the series $\sum_{k \geq 1} 2^{-k} \ln \omega_k^{-1}$ converges.

According to Lemma 10.5, the diffeomorphism $\Phi(\gamma)$ is formally linearizable. According to Lemma 10.6, all eigenvalues $(\lambda_1, \ldots, \lambda_m)$ of the differential $D(\gamma)$ of $\Phi(\gamma)$ at the origin are algebraic numbers. An important theorem of Baker shows that there is a constant $C > 0$ such that for all integers k, we have $\omega_k \geq \exp(-Ck)$ [3, Theorem 3.1]. It follows that the condition (Ω) is satisfied and one can apply Brjuno's theorem. □

REMARK 10.8. In most cases, the spectrum of $D(\gamma)$ contains many resonances. Not only the determinant of $D(\gamma)$ is one since there is no non-trivial homomorphisms from Γ to $\mathbf{R}$ but there are extra resonances coming from the structure of linear representations. Suppose for example that $\Gamma = SL(n, \mathbf{Z})$ and that $\Phi = D$ is the restriction to Γ of a linear representation of $SL(n, \mathbf{R})$ in $GL(m, \mathbf{R})$. Then the many integral linear relations between the weights of this representation provide corresponding multiplicative relations between the eigenvalues of the matrix $D(\gamma)$. Hence, in order to prove the previous lemma, the classical linearization theorem of Siegel is not sufficient ([28]): one has to use the more powerful theorem of Brjuno which allows resonances but it was indeed necessary to first prove the formal linearizability.

Of course, our problem now is that the diffeomorphisms which linearize the $\Phi(\gamma)$ might depend on γ. The difficulty comes again from the resonances since these imply that the centralizers of $D(\gamma)$ are big inside the group of analytic diffeomorphisms.

Denote by $\mathrm{Diff}(\mathbf{R}^m, 0)$ the group of germs of real analytic diffeomorphisms of $\mathbf{R}^m$ at 0 and by $\widehat{\mathrm{Diff}}(\mathbf{R}^m, 0)$ the group of formal diffeomorphisms. We can consider Φ as a homomorphism from Γ to $\mathrm{Diff}(\mathbf{R}^m, 0) \subset \widehat{\mathrm{Diff}}(\mathbf{R}^m, 0)$. The linear part D of Φ is a homomorphism from Γ to $GL(m, \mathbf{R})$.

We can assume that $D(\Gamma)$ is infinite. Indeed, if $D(\Gamma)$ is finite, the kernel of D acts trivially by Thurston's theorem so that the action Φ factors through a finite group and is therefore linearizable.

By Lemma 10.5, there is an element $\widehat{f}$ in $\widehat{\mathrm{Diff}}(\mathbf{R}^m, 0)$ which conjugates Φ and D. Let $H \subset GL(m, \mathbf{R})$ be the Zariski closure of $D(\Gamma)$. According to [27, ibid.], H is a semi-simple group. Let $\phi\colon H \to \widehat{\mathrm{Diff}}(\mathbf{R}^m, 0)$ be defined by $\phi(h) = \widehat{f} h \widehat{f}^{-1}$ so that for $\gamma \in \Gamma$, we have $\Phi(\gamma) = \phi\big(D(\gamma)\big)$. If we could show that $\phi(H) \subset \mathrm{Diff}(\mathbf{R}^m, 0)$ then we could apply Kushnirenko's theorem and there would exist an element f of $\mathrm{Diff}(\mathbf{R}^m, 0)$ such that $f\phi(H)f^{-1}$ is contained in $GL(m, \mathbf{R})$. Since $f\Phi(\gamma)f^{-1} = f\phi\big(D(\gamma)\big)f^{-1}$ the convergent diffeomorphism f would linearize $\Phi(\Gamma)$ as required.

Therefore, we denote by $H_0 \subset H$ the inverse image of $\mathrm{Diff}(\mathbf{R}^m, 0)$ by ϕ and we shall show that $H_0 = H$. Observe first that obviously $D(\Gamma)$ is contained in H_0 since $\phi\big(D(\gamma)\big) = \Phi(\gamma)$ is convergent by hypothesis.

For each γ in Γ, denote by $\langle D(\gamma)\rangle$ the Zariski closure of the group generated by $D(\gamma)$ in $GL(m, \mathbf{R})$. We claim that $\langle D(\gamma)\rangle$ is contained in H_0 if $D(\gamma)$ is semi-simple.

Indeed, by Lemma 10.7, we know that there is a convergent diffeomorphism f_γ such that $f_\gamma \Phi(\gamma) f_\gamma^{-1} = D(\gamma)$. The algebraic group consisting of those elements g of $GL(m, \mathbf{R})$ such that $f_\gamma \phi(g) f_\gamma^{-1} = g$ contains $D(\gamma)$, hence $\langle D(\gamma)\rangle$. It follows that every element of $\langle D(\gamma)\rangle$ has an image under ϕ which is conjugate by f_γ to a linear map so that in particular $\phi(\langle D(\gamma)\rangle)$ consists of convergent diffeomorphisms and $\langle D(\gamma)\rangle$ is indeed contained in H_0 as we claimed.

Observe that by Remark 2.1 we can replace Γ by a subgroup of finite index. In particular, using Selberg's lemma, we can assume that $D(\Gamma)$ is torsion free and, more precisely, that if some power of some $D(\gamma)$ lies in a normal subgroup of H then $D(\gamma)$ is in this subgroup (note that there are finitely many such normal subgroups).

Since $D(\gamma)$ has infinite order (if γ is non-trivial), $\langle D(\gamma)\rangle$ has positive dimension so that it contains a non-trivial one-parameter group. Hence every non trivial semi-simple element in $D(\Gamma)$ yields a one-parameter group contained in H_0. We now show that these one-parameter subgroups generate the connected component of the identity in H. Observe the following elementary fact: if a family of vectors spans the Lie algebra of a Lie group, then the one-parameter groups generated by these vectors generate the connected component of the identity. Therefore, we consider the linear span $\mathfrak{C}$ in the Lie algebra $\mathfrak{H}$ of H of the Lie algebras of all the subgroups $\langle D(\gamma)\rangle$ for γ semi-simple. It is enough to show that $\mathfrak{C} = \mathfrak{H}$. Note that $\mathfrak{C}$ is certainly non-trivial since semi-simple elements are Zariski dense in H. Note also that $\mathfrak{C}$ is invariant under the adjoint action of $D(\Gamma)$, hence under the adjoint action of H since $D(\Gamma)$ is Zariski dense in H. It follows that $\mathfrak{C}$ coincides with the product of some of the simple factors of $\mathfrak{H}$. The only possibility is that $\mathfrak{C} = \mathfrak{H}$ since otherwise, all the semi-simple $D(\gamma)$ would have some power contained in the same product of some but not all of the simple factors of H (note that the algebraic Abelian group $\langle D(\gamma)\rangle$ has a finite number of connected components). This implies that all semi-simple elements of $D(\Gamma)$ are contained in some non trivial normal subgroup of H. This is not possible by the following argument. In the algebraic group H, there is a non-empty open Zariski set consisting of semi-simple elements which are not contained in any non-trivial normal subgroup of H. Since $D(\Gamma)$ is Zariski dense in H, it intersects non-trivially this open set.

It follows that H_0 contains the connected component of the identity of H. Therefore H_0 is a semi-simple Lie group of finite index in H. By Kushnirenko's theorem, we can analytically linearize $\phi(H)$ (one also uses Remark 2.1) and in particular $\Phi(\Gamma)$.

Theorem 10.4 is proved.

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