1. Introduction

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **43 (1997)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **12.07.2024**

L'Enseignement Mathématique, t. 43 (1997), p. 173–198

POLYGON SPACES AND GRASSMANNIANS

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ABSTRACT. We study the moduli spaces of polygons in $\mathbf{R}^2$ and $\mathbf{R}^3$, identifying them with subquotients of 2-Grassmannians using a symplectic version of the Gel′fand-MacPherson correspondence. We show that the bending flows defined by Kapovich-Millson arise as a reduction of the Gel′fand-Cetlin system on the Grassmannian, and with these determine the pentagon and hexagon spaces up to equivariant symplectomorphism. Other than invocation of Delzant's theorem, our proofs are purely polygon-theoretic in nature.

1. INTRODUCTION

Let ${}^m\widetilde{\mathcal{P}}^k$ be the space of m-gons in $\mathbf{R}^k$ up to translation and positive homotheties (precise definitions in §2). This space comes with several structures: an action of $O(k)$, an action of S_m permuting the edges, and a function $\ell : {}^m\widetilde{\mathcal{P}}^k \longrightarrow \mathbf{R}^m$ taking a polygon ρ to the lengths of its edges (once the perimeter of ρ is fixed). The quotients of ${}^m\widetilde{\mathcal{P}}^k$ by SO_k (or O_k) are the moduli spaces ${}^m\mathcal{P}^k_+$ (respectively, ${}^m\mathcal{P}^k$). Fixing a reflection in $O(k)$ provides an involution on ${}^m\widetilde{\mathcal{P}}^k$ and ${}^m\mathcal{P}^k_+$ whose fixed point sets are ${}^m\widetilde{\mathcal{P}}^{k-1}$ and ${}^m\mathcal{P}^{k-1}$. The goal of this paper is to understand the topology of these various spaces and the geometric structures that they naturally carry when $k = 2$ or 3. They are closely related to more familiar objects (Grassmannians, projective spaces, Hopf bundles, etc.) The spaces ${}^m\mathcal{P}^k(\alpha) := \ell^{-1}(\alpha)$ of polygons with given side-lengths $\alpha \in \mathbf{R}^m$ are of particular interest.

The great miracle occurs when $k = 3$, because $\mathbf{R}^3$ is isomorphic to the space $I\mathbf{H}$ of pure imaginary quaternions, and the 2-sphere in $\mathbf{R}^3$ is Kähler. The tools of symplectic geometry can then be used. Most prominent is a

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*) Both authors thank the Fonds National Suisse de la Recherche Scientifique for its support.

symplectic version of the Gel′fand-MacPherson correspondence identifying the spaces ${}^m\mathcal{P}^3(\alpha)$ as symplectic quotients of the Grassmannian of 2-planes in $\mathbf{C}^m$. Earlier occurrences of symplectic geometry in the study of polygon spaces can be found in [Kl] and [KM2].

While this paper illustrates many phenomena in symplectic geometry, the proofs are entirely polygon-theoretic and involve only classical differential topology. Nonetheless, many of the examples are new, interesting in their own right and instructive for both fields.

Among our results:

1. The identification of the polygon space ${}^m\mathcal{P}^3$ with $\mathbf{G}_2(\mathbf{C}^m)/\big(U(1)^m\big)$ intertwines complex conjugation on the complex Grassmannian (with fixed point set the real Grassmannian) and spatial reflection on the polygon moduli space (with fixed point set planar polygons). The fact that 3-dimensional and planar polygons have the same allowed values of ℓ is then an illustration of a theorem of Duistermaat ([Du]). (As is always true, and yet always mysterious, it is helpful for studying the real case — here planar polygons — to extend to the complex case — here polygons in $\mathbf{R}^3$.)

2. Identification of the densely defined “bending flows” ([Kl] and [KM2]) on the polygon spaces with the reduction of the Gel′fand-Cetlin system [GS1] on the Grassmannian.

3. In some cases, the bending flows are globally defined, and by Delzant's reconstruction theory the spaces are equivariantly symplectomorphic to toric varieties (for instance when $m \leq 6$, as noted in [KM2]). We give a precise description of the moment polytope and so explicitly identify the toric varieties.

Contrary to the usual custom in symplectic reduction, it turned out here to be more natural to take symplectic quotients by *first* quotienting the original manifold by the group, and to *then* pick out a symplectic leaf of the resulting Poisson space — the intermediate quotient spaces all have natural polygon-theoretic interpretations. However, they are never complex; readers wishing a more geometric-invariant-theoretic construction of these spaces should look at [KM2].

This paper is structured as follows. Section 2 gives the definitions and elementary properties of polygon spaces. Sections 3 and 4 relate them to Grassmannians, and prove some facts about the moment map for the torus action on the Grassmannian by polygon-theoretic means. In section 4 is also calculated the exact relation between the Kähler structures in this paper and the ones in [KM2]. Section 5 relates the “bending flows” of [Kl] and [KM2] with the Gel′fand-Cetlin system on the Grassmannian. Section 6 uses this to calculate the quadrilateral, pentagon and hexagon spaces. Section 7 lists some open problems.

The study of the polygon spaces will be pursued in a forthcoming paper [HK] in which we shall compute the cohomology ring of these spaces.

The first author was incited by Sylvain Cappell to introduce symplectic geometry in his study of polygon spaces. He is also grateful to Lisa Jeffrey and Michèle Audin for useful conversations. The two authors started this work at the workshop in symplectic geometry organized in Cambridge by the Isaac Newton Institute (Fall 1994). The second author would like to thank Richard Montgomery for teaching him about dual pairs, and Michael Thaddeus for pointing out the link to moduli spaces of flat connections; also the University of Geneva for its hospitality while this paper was being written.

2. THE POLYGON SPACES

(2.1) Let V be a real vector space and m a positive integer. Let ${}^m\mathcal{F}(V)$ be the real vector space of all maps $\rho\colon \{1,2,\ldots,m\} \to V$ such that $\sum_{j=1}^m \rho(j) = 0$. An element $\rho \in {}^m\mathcal{F}(V)$ will be regarded as a closed polygonal path in V

$$0 \bullet\!\!\rightarrow\!\!\bullet\ \rho(1) \bullet\!\!\rightarrow\!\!\bullet\ \rho(1)+\rho(2) \bullet\!\!\rightarrow\!\!\bullet \cdots \bullet\!\!\rightarrow\!\!\bullet \sum_{j=1}^m \rho(j) = 0$$

of m steps, or, alternately, as a configuration in V (up to translation) of a polygon of m sides. We shall call an element $\rho \in {}^m\mathcal{F}(V)$ an *m-polygon (in V)* and a *proper polygon* when $\rho(j) \neq 0\ \forall j$. We use the notation ${}^m\mathcal{F}^k$ for the space ${}^m\mathcal{F}(\mathbf{R}^k)$.

The group $\mathbf{R}_+$ of positive homotheties of V acts freely and properly on ${}^m\mathcal{F}(V) - \{0\}$. The quotient ${}^m\widetilde{\mathcal{P}}(V) := ({}^m\mathcal{F}(V) - \{0\})/\mathbf{R}_+$ then inherits a structure of smooth manifold diffeomorphic to a sphere. For instance, ${}^m\widetilde{\mathcal{P}}^k := ({}^m\mathcal{F}^k - \{0\})/\mathbf{R}_+$ is diffeomorphic to the sphere $S^{k(m-1)-1}$.

(2.2) Suppose now that V is oriented and is a Euclidean space, namely V is endowed with a scalar product. The group $O(V)$ of isometries of V acts on ${}^k\mathcal{F}^m$ and ${}^m\widetilde{\mathcal{P}}(V)$; we define the moduli spaces

$${}^m\mathcal{P}(V)_+ := SO(V)\backslash{}^m\widetilde{\mathcal{P}}(V) \quad \text{and} \quad {}^m\mathcal{P}(V) := O(V)\backslash{}^m\widetilde{\mathcal{P}}(V)$$

of m-polygons in V, up to similitude (where $SO(V)$ is the identity component of $O(V)$). Observe that any orientation preserving isometry $h\colon V \xrightarrow{\sim} \mathbf{R}^k$ produces identifications