

## 2. The polygon spaces

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calculate the quadrilateral, pentagon and hexagon spaces. Section 7 lists some open problems.

The study of the polygon spaces will be pursued in a forthcoming paper [HK] in which we shall compute the cohomology ring of these spaces.

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## 2. THE POLYGON SPACES

(2.1) Let  $V$  be a real vector space and  $m$  a positive integer. Let  ${}^m\mathcal{F}(V)$  be the real vector space of all maps  $\rho: \{1, 2, \dots, m\} \rightarrow V$  such that  $\sum_{j=1}^m \rho(j) = 0$ . An element  $\rho \in {}^m\mathcal{F}(V)$  will be regarded as a closed polygonal path in  $V$

$$0 \bullet \rightarrow \bullet \rho(1) \bullet \rightarrow \bullet \rho(1) + \rho(2) \bullet \rightarrow \bullet \dots \bullet \rightarrow \bullet \sum_{j=1}^m \rho(j) = 0$$

of  $m$  steps, or, alternately, as a configuration in  $V$  (up to translation) of a polygon of  $m$  sides. We shall call an element  $\rho \in {}^m\mathcal{F}(V)$  an  $m$ -*polygon* (in  $V$ ) and a *proper polygon* when  $\rho(j) \neq 0 \forall j$ . We use the notation  ${}^m\mathcal{F}^k$  for the space  ${}^m\mathcal{F}(\mathbf{R}^k)$ .

The group  $\mathbf{R}_+$  of positive homotheties of  $V$  acts freely and properly on  ${}^m\mathcal{F}(V) - \{0\}$ . The quotient  ${}^m\tilde{\mathcal{P}}(V) := ({}^m\mathcal{F}(V) - \{0\})/\mathbf{R}_+$  then inherits a structure of smooth manifold diffeomorphic to a sphere. For instance,  ${}^m\tilde{\mathcal{P}}^k := ({}^m\mathcal{F}^k - \{0\})/\mathbf{R}_+$  is diffeomorphic to the sphere  $S^{k(m-1)-1}$ .

(2.2) Suppose now that  $V$  is oriented and is a Euclidean space, namely  $V$  is endowed with a scalar product. The group  $O(V)$  of isometries of  $V$  acts on  ${}^k\mathcal{F}^m$  and  ${}^m\tilde{\mathcal{P}}(V)$ ; we define the moduli spaces

$${}^m\mathcal{P}(V)_+ := SO(V) \backslash {}^m\tilde{\mathcal{P}}(V) \quad \text{and} \quad {}^m\mathcal{P}(V) := O(V) \backslash {}^m\tilde{\mathcal{P}}(V)$$

of  $m$ -polygons in  $V$ , up to similitude (where  $SO(V)$  is the identity component of  $O(V)$ ). Observe that any orientation preserving isometry  $h: V \xrightarrow{\sim} \mathbf{R}^k$  produces identifications

$${}^m\mathcal{P}(V)_+ \simeq {}^m\mathcal{P}_+^k := SO_k \backslash {}^m\tilde{\mathcal{P}}^k \quad \text{and} \quad {}^m\mathcal{P}(V) \simeq {}^m\mathcal{P}^k := O_k \backslash {}^m\tilde{\mathcal{P}}^k.$$

We shall use the fact that these identifications do not depend on the choice of  $h$  and thus  ${}^m\mathcal{P}(V)_+$  and  ${}^m\mathcal{P}(V)$ , for any Euclidean space  $V$ , are canonically identified with  ${}^m\mathcal{P}_+^k$  and  ${}^m\mathcal{P}^k$ .

(2.3) The “degree of improperness” of polygons provides a stratification

$$\emptyset = E_1 {}^m\tilde{\mathcal{P}}(V) \subset E_2 {}^m\tilde{\mathcal{P}}(V) \subset \cdots \subset E_{m-1} {}^m\tilde{\mathcal{P}}(V) \subset E_m {}^m\tilde{\mathcal{P}}(V) = {}^m\tilde{\mathcal{P}}(V)$$

where

$$E_j {}^m\tilde{\mathcal{P}}(V) := \{\rho \in {}^m\tilde{\mathcal{P}}(V) \mid \#\{s \mid \rho(s) = 0\} \geq m - j\}.$$

The “open stratum”  $E_j {}^m\tilde{\mathcal{P}}(V) - E_{j-1} {}^m\tilde{\mathcal{P}}(V)$  is a smooth submanifold of  ${}^m\tilde{\mathcal{P}}(V)$  of dimension  $(j-1)k - 1$  if  $k = \dim V$ . The top open stratum  ${}^m\tilde{\mathcal{P}}(V) - E_{m-1} {}^m\tilde{\mathcal{P}}(V)$ , open and dense in  ${}^m\tilde{\mathcal{P}}(V)$ , is the space of proper polygons.

As this stratification is  $O(V)$ -invariant, it projects onto stratifications  $\{E_j {}^m\mathcal{P}_+^k\}$  and  $\{E_j {}^m\mathcal{P}^k\}$  of the moduli spaces (using the canonical identifications of (2.2)). We shall see in §3 that the above stratifications describe the singular loci of smooth orbifold structures on the spaces  ${}^m\tilde{\mathcal{P}}(V)$ ,  ${}^m\mathcal{P}_+^k$  and  ${}^m\mathcal{P}^k$ .

(2.4) The map  $\rho \mapsto |\rho| := \sum_{j=1}^m |\rho(j)|$  which associates to a polygon  $\rho$  its total perimeter is a norm on  ${}^m\mathcal{F}(V)$ . We denote by  $S({}^m\mathcal{F}(V))$  the sphere of radius 2 for this norm. Each class in  ${}^m\tilde{\mathcal{P}}(V)$  has a unique representative in  $S({}^m\mathcal{F}(V))$  which gives a topological embedding  $\iota: {}^m\tilde{\mathcal{P}}(V) \rightarrow S({}^m\mathcal{F}(V))$  whose image is  $S({}^m\mathcal{F}(V))$ . The image by  $\iota$  of  $E_{m-1} {}^m\tilde{\mathcal{P}}(V)$  is the subset of  $S({}^m\mathcal{F}(V))$  where  $S({}^m\mathcal{F}(V))$  fails to be a smooth submanifold of  ${}^m\mathcal{F}(V)$ . However, the restriction of  $\iota$  to each  $E_j {}^m\tilde{\mathcal{P}}(V) - E_{j-1} {}^m\tilde{\mathcal{P}}(V)$  is a smooth embedding.

The map  $\tilde{\ell}: {}^m\mathcal{F}(V) \rightarrow \mathbf{R}^m$  defined by  $\tilde{\ell}(\rho) := (|\rho(1)|, \dots, |\rho(m)|)$  associates to a polygon its side-lengths. We define  $\ell: {}^m\tilde{\mathcal{P}}(V) \rightarrow \mathbf{R}^m$  by  $\ell := \tilde{\ell} \circ \iota$ . We shall also use the notation  $\ell_i(\rho)$  for  $|\rho(i)|$ . These maps are invariant under the  $O(V)$ -action and thus define maps (always called  $\ell$ )

$$\ell: {}^m\mathcal{P}_+^k \rightarrow \mathbf{R}^m \quad \text{and} \quad \ell: {}^m\mathcal{P}^k \rightarrow \mathbf{R}^m$$

which are smooth on each open stratum.

(2.5) Let  $\tau: V \rightarrow V$  be the orthogonal reflection through some hyperplane  $\Pi$  in  $V$ . One has the involution  $\rho \mapsto \check{\rho} := \tau \circ \rho$  on  ${}^m\mathcal{F}(V)$  and  ${}^m\tilde{\mathcal{P}}(V)$  whose fixed-point space is naturally  ${}^m\mathcal{F}(\Pi)$  and  ${}^m\tilde{\mathcal{P}}(\Pi)$ . If  $h \in SO(V)$ , one has

$$\tau \circ h = \underbrace{(\tau \circ h \circ \tau \circ h^{-1})}_{\in SO(V)} \circ h \circ \tau.$$

Hence the involution descends to an involution (still denoted  $\rho \mapsto \check{\rho}$ ) on  ${}^m\mathcal{P}_+^k$ . If  $\tau'$  is an orthogonal reflection with respect to another hyperplane  $\Pi'$ , then the formula  $\tau \circ \rho' = (\tau' \circ \tau) \circ \tau \circ \rho$  shows that the induced involution on  ${}^m\mathcal{P}_+^k$  does not depend on the choice of  $\tau$ . The fixed point space of  $\check{\cdot}$  is  ${}^m\mathcal{P}^{k-1}$ . Observe that  $\check{\rho} = \rho$  in  ${}^m\mathcal{P}^k$ .

EXAMPLES

(2.6) *Polygons in the line.* The space  ${}^m\tilde{\mathcal{P}}^1 = {}^m\tilde{\mathcal{P}}_+^1$  is diffeomorphic to the sphere  $S^{m-2}$ . Under this identification, the  $O_1$ -action becomes the antipodal map and thus  ${}^m\mathcal{P}^1$  is a smooth manifold diffeomorphic to  $\mathbf{R}P^{m-2}$ . For example,  ${}^3\tilde{\mathcal{P}}^1 \simeq S^1$  and  ${}^3\mathcal{P}^1 \simeq \mathbf{R}P^1$ . The stratum  $E_2{}^3\tilde{\mathcal{P}}^1$  consists of 3 pairs of antipodal points and thus  $E_2{}^3\mathcal{P}^1$  is a set of 3 points, the three triangles with one side of length 0. This corresponds to the fact that  $S({}^3\mathcal{F}^1)$  is a regular hexagon and  $O_1 \backslash S({}^3\mathcal{F}^1)$  is a triangle. Actually, the map  $\ell : {}^3\mathcal{P}^1 \rightarrow \mathbf{R}^3$  produces homeomorphisms

$${}^3\mathcal{P}^1 \xrightarrow[\simeq]{\ell} \{(x, y, z) \in \mathbf{R}_{\geq 0}^3 \mid x + y + z = 2 \text{ and } \pm x \pm y \pm z = 0\}.$$

(2.7) *Polygons in the plane.* Identifying  $\mathbf{R}^2$  with  $\mathbf{C}$ , the space  ${}^m\mathcal{F}^2$  is a complex vector space isomorphic to  $\mathbf{C}^{m-1}$  and the (free)  $SO_2$ -action corresponds to the diagonal  $U_1$ -action. As in (2.6) one establishes the diffeomorphisms

$$\begin{array}{ccc} {}^m\tilde{\mathcal{P}}^2 & \xrightarrow{\simeq} & S^{2m-3} \\ \downarrow & & \downarrow \\ {}^m\mathcal{P}_+^2 & \xrightarrow{\simeq} & \mathbf{C}P^{m-2} \end{array}$$

The above diffeomorphisms conjugate the involutions  $\check{\cdot}$  with the complex conjugations of  $\mathbf{C}^{m-1}$  and  $\mathbf{C}P^{m-2}$ . Also, the involution  $\check{\cdot}$  on  ${}^m\mathcal{P}_+^2$  coincides with the residual  $O_1$  action and therefore  ${}^m\mathcal{P}^2$  is the quotient of  $\mathbf{C}P^{m-2}$  by its complex conjugation.

For example,  ${}^3\tilde{\mathcal{P}}^2$ , the space of planar triangles, is diffeomorphic to the sphere  $S^3$ . The singular stratum  $E_2({}^3\tilde{\mathcal{P}}^1)$  is a link of three circles which are  $SO_2$ -orbits (therefore, any two of them constitute a Hopf link). The quotient  ${}^3\mathcal{P}_+^2$  is identified with  $\mathbf{C}P^1$  and  $E_2({}^3\tilde{\mathcal{P}}_+^1)$  is a set of three points in  $\mathbf{C}P^1$ .

Finally,  ${}^3\mathcal{P}^2 \simeq \mathbf{C}P^1 / \{z \sim \bar{z}\}$  is homeomorphic, via the length-side map  $\ell$ , to the solid triangle

$${}^3\mathcal{P}^2 = {}^3\mathcal{P}^3 \xrightarrow[\simeq]{\ell} \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_1 + x_2 + x_3 = 2 \text{ and } 0 \leq x_i \leq 1\}$$

with boundary  ${}^3\mathcal{P}^1$ .

### 3. QUATERNIONS, GRASSMANNIANS AND STRUCTURES ON THE FULL POLYGON SPACES

(3.1) Let  $\mathbf{H} = \mathbf{C} \oplus \mathbf{C}j$  be the skew-field of quaternions; the space  $I\mathbf{H}$  of pure imaginary quaternions is equipped with the orthonormal basis  $i, j$  and  $k = ij$ , giving rise to an isometry with  $\mathbf{R}^3$  which turns the pure imaginary part of the quaternionic multiplication  $pq$  into the usual cross product  $p \times q$ . The space  ${}^m\mathcal{F}^3$  is thus identified with  ${}^m\mathcal{F}(I\mathbf{H})$  which gives rise to the canonical identifications on the the various moduli spaces (see (2.2)).

Recall that the correspondence

$$\eta : u + vj \mapsto \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$$

gives an injective  $\mathbf{R}$ -algebra homomorphism  $\eta : \mathbf{H} \longrightarrow \mathcal{M}_{(2 \times 2)}(\mathbf{C})$ . This enables a matrix  $P \in U_2$  to act on the right or on the left on  $\mathbf{H}$ . It also identifies the group  $S^3$  of unit quaternions with  $SU_2$ .

(3.2) The Hopf map  $\phi : \mathbf{H} \longrightarrow I\mathbf{H}$  defined by

$$\phi(q) := \bar{q} i q$$

sends the 3-sphere of radius  $\sqrt{r}$  in  $\mathbf{H}$  onto the 2-sphere of radius  $r$  in  $I\mathbf{H}$ . (The formulae given in the original paper by Hopf [Ho, §5] actually correspond to the map  $q \mapsto \bar{q}kq$ .) The equality  $\phi(q) = \phi(q')$  occurs if and only if  $q' = e^{i\theta} q$ . The map  $\phi$  satisfies the equivariance relation  $\phi(q \cdot P) = P^{-1} \cdot \phi(q) \cdot P$ . Writing  $q = u + vj$  with  $u, v \in \mathbf{C}$ , one has

$$\phi(u + vj) = (\bar{u} - j\bar{v}) i(u + vj) = i(\bar{u} + j\bar{v})(u + vj) = i[ (|u|^2 - |v|^2) + 2\bar{u}vj ].$$

(3.3) Observe that if  $q = s + tj$  with  $s, t \in \mathbf{R}$ , then  $\phi(q) = iq^2$ . This plane  $\mathbf{R} \oplus \mathbf{R}j$  of its images is the fixed point set of the involution  $a + bj \mapsto \bar{a} + \bar{b}j$  that will be used later. Its image under  $\phi$  is  $\mathbf{R}i \oplus \mathbf{R}k$ .

(3.4) REMARK.  $I\mathbf{H}$ , with the Lie bracket  $[p, q] = pq - qp = 2 \operatorname{Im}(pq)$ , is the Lie algebra for the group  $U_1(\mathbf{H}) \simeq SU_2 \simeq S^3$ . The pairing