

# 3. Quaternions, Grassmannians and structures on the full polygon spaces

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Finally,  ${}^3\mathcal{P}^2 \simeq \mathbf{C}P^1 / \{z \sim \bar{z}\}$  is homeomorphic, via the length-side map  $\ell$ , to the solid triangle

$${}^3\mathcal{P}^2 = {}^3\mathcal{P}^3 \xrightarrow[\simeq]{\ell} \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_1 + x_2 + x_3 = 2 \text{ and } 0 \leq x_i \leq 1\}$$

with boundary  ${}^3\mathcal{P}^1$ .

### 3. QUATERNIONS, GRASSMANNIANS AND STRUCTURES ON THE FULL POLYGON SPACES

(3.1) Let  $\mathbf{H} = \mathbf{C} \oplus \mathbf{C}j$  be the skew-field of quaternions; the space  $I\mathbf{H}$  of pure imaginary quaternions is equipped with the orthonormal basis  $i, j$  and  $k = ij$ , giving rise to an isometry with  $\mathbf{R}^3$  which turns the pure imaginary part of the quaternionic multiplication  $pq$  into the usual cross product  $p \times q$ . The space  ${}^m\mathcal{F}^3$  is thus identified with  ${}^m\mathcal{F}(I\mathbf{H})$  which gives rise to the canonical identifications on the various moduli spaces (see (2.2)).

Recall that the correspondence

$$\eta : u + vj \mapsto \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$$

gives an injective  $\mathbf{R}$ -algebra homomorphism  $\eta : \mathbf{H} \longrightarrow \mathcal{M}_{(2 \times 2)}(\mathbf{C})$ . This enables a matrix  $P \in U_2$  to act on the right or on the left on  $\mathbf{H}$ . It also identifies the group  $S^3$  of unit quaternions with  $SU_2$ .

(3.2) The Hopf map  $\phi : \mathbf{H} \longrightarrow I\mathbf{H}$  defined by

$$\phi(q) := \bar{q} i q$$

sends the 3-sphere of radius  $\sqrt{r}$  in  $\mathbf{H}$  onto the 2-sphere of radius  $r$  in  $I\mathbf{H}$ . (The formulae given in the original paper by Hopf [Ho, §5] actually correspond to the map  $q \mapsto \bar{q}kq$ .) The equality  $\phi(q) = \phi(q')$  occurs if and only if  $q' = e^{i\theta} q$ . The map  $\phi$  satisfies the equivariance relation  $\phi(q \cdot P) = P^{-1} \cdot \phi(q) \cdot P$ . Writing  $q = u + vj$  with  $u, v \in \mathbf{C}$ , one has

$$\phi(u + vj) = (\bar{u} - j\bar{v}) i (u + vj) = i(\bar{u} + j\bar{v})(u + vj) = i[ (|u|^2 - |v|^2) + 2\bar{u}vj ].$$

(3.3) Observe that if  $q = s + tj$  with  $s, t \in \mathbf{R}$ , then  $\phi(q) = iq^2$ . This plane  $\mathbf{R} \oplus \mathbf{R}j$  of its images is the fixed point set of the involution  $a + bj \mapsto \bar{a} + \bar{b}j$  that will be used later. Its image under  $\phi$  is  $\mathbf{R}i \oplus \mathbf{R}k$ .

(3.4) REMARK.  $I\mathbf{H}$ , with the Lie bracket  $[p, q] = pq - qp = 2 \operatorname{Im}(pq)$ , is the Lie algebra for the group  $U_1(\mathbf{H}) \simeq SU_2 \simeq S^3$ . The pairing

$(q, q') \mapsto -\operatorname{Re}(qq') = \langle q, q' \rangle$  identifies  $I\mathbf{H}$  with its dual. If  $\mathbf{H} \simeq \mathbf{C} \oplus \mathbf{C}$  is endowed with the standard Kähler form, then the map  $\frac{1}{2}\phi$  is the moment map for the Hamiltonian action of  $U_1(\mathbf{H})$  on  $\mathbf{H}$  (the factor  $\frac{1}{2}$  can be checked by restricting the action to the  $S^1$ -action on  $\mathbf{C}$ ).

(3.5) Let  $\mathbf{V}_2(\mathbf{C}^m)$  be the space of  $(m \times 2)$ -matrices

$$(a, b) := \begin{pmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_m & b_m \end{pmatrix} \in \mathcal{M}_{m \times 2}(\mathbf{C})$$

such that  $|a| = |b| = 1$  and  $\langle a, b \rangle = 0$ .  $\mathbf{V}_2(\mathbf{C}^m)$  is the Stiefel manifold of orthonormal 2-frames in  $\mathbf{C}^m$ . The group  $U_m$  acts transitively on the left on  $\mathbf{V}_2(\mathbf{C}^m)$  producing the diffeomorphism  $\mathbf{V}_2(\mathbf{C}^m) = U_m/U_{m-2}$ . One has the conjugation on  $\mathbf{V}_2(\mathbf{C}^m)$  given by  $(a, b) \mapsto (\bar{a}, \bar{b})$  with fixed-point space the Stiefel manifold  $\mathbf{V}_2(\mathbf{R}^m) = O_m/O_{m-2}$  of orthonormal 2-frames in  $\mathbf{R}^m$ . Finally, the embedding  $\mathbf{V}_2(\mathbf{C}^m) \subset \mathbf{H}^m$  given by  $(a, b) \mapsto (\dots, a_r + b_r j, \dots)$  intertwines the conjugation on  $\mathbf{V}_2(\mathbf{C}^m)$  with the involution of (2.5) on  $\mathbf{H}^m$ . One thus gets an embedding  $\mathbf{V}_2(\mathbf{R}^m) \subset (\mathbf{R} \oplus \mathbf{R}j)^m$ .

Using the Hopf map  $\phi$  of (3.2), one defines the smooth map  $\Phi : \mathbf{V}_2(\mathbf{C}^m) \rightarrow {}^m\mathcal{F}(I\mathbf{H}) \simeq {}^m\mathcal{F}^3$  by the formula

$$\Phi(a, b) := (\phi(a_1 + b_1 j), \phi(a_2 + b_2 j), \dots, \phi(a_m + b_m j)).$$

The fact that  $\sum \phi(a_r + b_r j) = 0$  is equivalent to  $\langle a, b \rangle = 0$  and  $|a| = |b|$ . As  $|a| = |b| = 1$ , the image of  $\Phi$  is exactly  $S({}^m\mathcal{F}^3)$ . By composing with the projection  ${}^m\mathcal{F}^3 - \{0\} \rightarrow {}^m\tilde{\mathcal{P}}^3$ , one gets a surjective smooth map  $\Phi : \mathbf{V}_2(\mathbf{C}^m) \rightarrow {}^m\tilde{\mathcal{P}}^3$ . One checks that  $\Phi(a, b) = \Phi(a', b')$  if and only if  $(a, b)$  and  $(a', b')$  are in the same orbit under the action of the maximal torus  $U_1^m$  of diagonal matrices in  $U_m$ . This action is free when none of the  $(a_i, b_i)$ 's vanishes, namely if and only if  $\Phi(a, b)$  is a proper polygon. As  $\Phi(\bar{a}, \bar{b}) = \Phi(a, b)^\vee$ , the restriction of  $\Phi$  to the fixed points gives a smooth map  $\Phi_{\mathbf{R}} : \mathbf{V}_2(\mathbf{R}^m) \rightarrow {}^m\tilde{\mathcal{P}}(\mathbf{R}i \oplus \mathbf{R}k) \simeq {}^m\tilde{\mathcal{P}}^2$  with analogous properties. We have thus proved

**THEOREM 3.6.** *a) The smooth map  $\Phi : \mathbf{V}_2(\mathbf{C}^m) \rightarrow {}^m\tilde{\mathcal{P}}^3$  induces a homeomorphism  $\hat{\Phi} : U_1^m \backslash \mathbf{V}_2(\mathbf{C}^m) \xrightarrow{\simeq} {}^m\tilde{\mathcal{P}}^3$  such that  $\hat{\Phi}(\bar{a}, \bar{b}) = \Phi(a, b)^\vee$ . The restriction of  $\Phi$  above the space of proper polygons is a smooth principal  $U_1^m$ -bundle.*

*b) The smooth map  $\Phi_{\mathbf{R}} : \mathbf{V}_2(\mathbf{R}^m) \rightarrow {}^m\tilde{\mathcal{P}}^2$  induces a homeomorphism  $\hat{\Phi}_{\mathbf{R}} : O_1^m \backslash \mathbf{V}_2(\mathbf{R}^m) \xrightarrow{\simeq} {}^m\tilde{\mathcal{P}}^2$ . The restriction of  $\Phi_{\mathbf{R}}$  above the space of proper planar polygons is a principal  $O_1^m$ -covering.*

COROLLARY 3.7.  ${}^m\tilde{\mathcal{P}}^3 \simeq U_1^m \backslash U_m / U_{m-2}$  and  ${}^m\tilde{\mathcal{P}}^2 \simeq O_1^m \backslash O_m / O_{m-2}$ .

(3.8) Let  $\mathbf{G}_2(\mathbf{C}^m)$  be the Grassmann manifold of 2-planes in  $\mathbf{C}^m$ . The map  $\mathbf{V}_2(\mathbf{C}^m) \rightarrow \mathbf{G}_2(\mathbf{C}^m)$  which associates to  $(a, b)$  the plane generated by  $a$  and  $b$  is the projection  $\mathbf{V}_2(\mathbf{C}^m) \rightarrow \mathbf{V}_2(\mathbf{C}^m)/U_2$  (a principal  $U_2$  bundle), for the natural right action of  $U_2$  on  $\mathbf{V}_2(\mathbf{C}^m) \subset \mathcal{M}_{m \times 2}(\mathbf{C})$ . This projection is  $U_m$ -equivariant, equivalent to the projection  $U_m/U_{m-2} \rightarrow U_m/U_2 \times U_{m-2}$ .

The map  $\Phi : \mathbf{V}_2(\mathbf{C}^m) \rightarrow {}^m\tilde{\mathcal{P}}^3$  satisfies

$$\Phi((a, b)P) = P^{-1} \Phi(a, b)P \quad \text{for } (a, b) \in \mathbf{V}_2(\mathbf{C}^m), P \in U_2.$$

The conjugation by  $P$  being an element of  $SO(I\mathbf{H})$ , one thus gets a map (still called  $\Phi$ ) from  $\mathbf{G}_2(\mathbf{C}^m)$  onto  ${}^m\mathcal{P}_+^3$ . The space  ${}^m\mathcal{P}_+^3$  has a smooth structure on the open-dense subset of non-lined polygons (which is where the  $SO_3$ -action was free) and, above this open-dense subset, the new map  $\Phi$  is smooth. The map  $\Phi$  intertwines the involutions and so restricts to a map  $\Phi_{\mathbf{R}} : \mathbf{G}_2(\mathbf{R}^m) \rightarrow {}^m\mathcal{P}^2$ , where  $\mathbf{G}_2(\mathbf{R}^m)$  is the Grassmannian of 2-planes in  $\mathbf{R}^m$ . In this case, an intermediate object is the Grassmannian  $\tilde{\mathbf{G}}_2(\mathbf{R}^m) = SO_m/SO_2 \times SO_{m-2}$  of oriented 2-planes in  $\mathbf{R}^m$  with the smooth map  $\Phi_{\mathbf{R}} \tilde{\mathbf{G}}_2(\mathbf{R}^m) \rightarrow {}^m\mathcal{P}_+^2 \simeq \mathbf{C}P^{m-2}$ . The action of  $U_1^m$  on  $\mathbf{V}_2(\mathbf{C}^m)$  descends to an action on  $\mathbf{G}_2(\mathbf{C}^m)$  which is no longer effective: its kernel is the diagonal subgroup  $\Delta$  of  $U_1^m$ , the center of  $U_m$ , isomorphic to  $U_1$ . The same holds true in the real case, replacing  $U_1$  by  $O_1$  (the diagonal subgroup of  $O_1^m$  is also denoted by  $\Delta$ ).

Using Theorem 3.6, the reader will easily prove the following

THEOREM 3.9. *a) The map  $\Phi : \mathbf{G}_2(\mathbf{C}^m) \rightarrow {}^m\mathcal{P}^3$  induces a homeomorphism  $\hat{\Phi} : U_1^m \backslash \mathbf{G}_2(\mathbf{C}^m) \xrightarrow{\simeq} {}^m\mathcal{P}^3$  such that  $\hat{\Phi}(\bar{a}, \bar{b}) = \Phi(a, b)^\vee$ . The restriction of  $\hat{\Phi}$  above the space of proper non-lined polygons is a smooth principal  $(U_1^m/\Delta)$ -bundle.*

*b) The smooth map  $\Phi_{\mathbf{R}} : \tilde{\mathbf{G}}_2(\mathbf{R}^m) \rightarrow {}^m\mathcal{P}_+^2$  induces a homeomorphism  $\hat{\Phi}_{\mathbf{R}} : O_1^m \backslash \tilde{\mathbf{G}}_2(\mathbf{R}^m) \xrightarrow{\simeq} {}^m\mathcal{P}_+^2$ . It is a smooth branched covering and, restricted above the space of proper polygons, a principal  $(O_1^m/\Delta)$ -covering.*

*c) The map  $\Phi_{\mathbf{R}} : \mathbf{G}_2(\mathbf{R}^m) \rightarrow {}^m\mathcal{P}^2$  induces a homeomorphism  $\hat{\Phi}_{\mathbf{R}} : O_1^m \backslash \mathbf{G}_2(\mathbf{R}^m) \xrightarrow{\simeq} {}^m\mathcal{P}^2$ . The restriction of  $\hat{\Phi}$  above the space of proper non-lined polygons is a principal  $(O_1^m/\Delta)$ -covering.*

COROLLARY 3.10. *One has homeomorphisms between the polygon spaces and the double cosets*

- a)  ${}^m\mathcal{P}^3 \simeq U_1^m \backslash U_m / (U_2 \times U_{m-2})$
- b)  ${}^m\mathcal{P}_+^2 \simeq S(O_1^m) \backslash SO_m / (SO_2 \times SO_{m-2})$ .
- c)  ${}^m\mathcal{P}^2 \simeq O_1^m \backslash O_m / (O_2 \times O_{m-2})$ .

(3.11) *Example.* As in (2.7) the example of planar triangles ( $m = 3$  and  $k = 2$ ) is interesting. The Stiefel manifold  $\mathbf{V}_2(\mathbf{R}^3)$  is diffeomorphic to the unit tangent bundle to  $S^2$ , in turn diffeomorphic to  $SO_3$ . The oriented Grassmannian  $\tilde{\mathbf{G}}_2(\mathbf{R}^3)$  can be identified with  $S^2$  by associating to an oriented plane its unit normal vector. The smooth map

$$\Phi_{\mathbf{R}} : S^2 \simeq \tilde{\mathbf{G}}_2(\mathbf{R}^3) \longrightarrow {}^3\mathcal{P}_+^2 \simeq S^2$$

is of degree 4, branched over the 3 points. This map can be visualized as follows: tessellate  $\mathbf{R}^2$  with equilateral triangles. Divide  $\mathbf{R}^2$  by the subgroup of isometries which preserve the tessellation and the orientation (it thus preserves a checkerboard coloring of the triangle tessellation). This quotient is a well known orbifold structure on  $S^2$  with three branched points. The projection  $\mathbf{R}^2 \longrightarrow S^2$  factors through an octahedron with a chess-board coloring of its faces. The residual map from this octahedron to  $S^2$  is our map  $\Phi_{\mathbf{R}}$ .

Take the pullback by  $\Phi_{\mathbf{R}}$  of the Hopf bundle  $S^3 \longrightarrow S^2$ . One gets a map of degree 4 from some lens space  $L$  onto  $S^3$ , with branched locus the link formed by three  $SO_2$ -orbits. The lens space will be doubly covered by  $SO_3$ . We thus get the map

$$\tilde{\Phi} : SO_3 \simeq \mathbf{V}_2(\mathbf{R}^3) \longrightarrow {}^3\tilde{\mathcal{P}}^2 \simeq S^3$$

of degree 8. Finally, one has  $\mathbf{G}_2(\mathbf{R}^3) \simeq \mathbf{RP}^2$  and  $\Phi_{\mathbf{R}}$  is the quotient of  $\mathbf{RP}^2$  by the action of  $O_1^3$  on each homogeneous coordinate. This quotient is a 2-simplex and one sees again that  ${}^3\mathcal{P}^2$  is a solid triangle.

(3.12) *Orbifold structures.* The maps  $\tilde{\Phi}_{\mathbf{R}}$  and  $\Phi_{\mathbf{R}}$  provide, for the spaces  ${}^2\tilde{\mathcal{P}}^2 \simeq S^{2m-3}$  and  ${}^m\mathcal{P}_+^2 \simeq \mathbf{CP}^{m-2}$ , a smooth orbifold structure. Each point has a neighbourhood homeomorphic to an open set of the quotient of  $(\mathbf{R}^2)^s$  by a subgroup of  $O_1^s$ , where  $O_1$  acts on each  $\mathbf{R}^2$  via the antipodal map. Observe that the map  $\Phi_{\mathbf{R}}$  is a “small cover” in the sense of [DJ]. The branched loci are  $E_{m-1} {}^m\tilde{\mathcal{P}}^2$  and  $E_{m-1} {}^m\mathcal{P}_+^2$  respectively. As for  ${}^m\mathcal{P}^2$  we have to add the branched locus  ${}^m\mathcal{P}^1$ . The generic points of  ${}^m\mathcal{P}^1$  have a neighbourhood modelled on the quotient of  $\mathbf{C}^{m-2}$  by complex conjugation.

Analogously, the map  $\Phi: \mathbf{G}_2(\mathbf{C}^m) \longrightarrow {}^m\mathcal{P}^3$  gives rise, for the space  ${}^m\tilde{\mathcal{P}}^3$ , to a smooth *complex orbifold structure*. By that we mean a space locally modelled on the quotient of  $\mathbf{C}^s$  by a subgroup of  $U_1^s$ . We define the space  $\mathcal{C}^\infty({}^m\mathcal{P}^3)$  of *smooth maps* from  ${}^m\mathcal{P}^3$  to the reals as the subspace of  $\mathcal{C}^\infty(\mathbf{G}_2(\mathbf{C}^m))$  which is invariant by the action of  $U_1^m$ .

(3.13) *Riemannian and Poisson structures.* Let  $\mathcal{H}(m)$  be the space of Hermitian  $(m \times m)$ -matrices, identified with  $\mathbf{u}_m^*$  via the pairing

$$\mathcal{H}(m) \times \mathbf{u}_m \longrightarrow \mathbf{R} \quad (H, X) \mapsto \frac{i}{2} \operatorname{tr}(HX).$$

This identification turns the co-adjoint action of  $U_m$  into the conjugation action on  $\mathcal{H}(m)$ . Consider the map  $\tilde{\Psi}: \mathcal{M}_{m \times 2}(\mathbf{C}) \longrightarrow \mathcal{H}(m)$  given by  $\tilde{\Psi}(a, b) := (a, b) \cdot (a, b)^*$ . One has  $\tilde{\Psi}(Q \cdot (a, b) \cdot P) = Q \cdot \tilde{\Psi}((a, b)) \cdot Q^*$  for  $P \in U_2$  and  $Q \in U_m$  and thus  $\mathcal{C} := \tilde{\Psi}(\mathbf{V}_2(\mathbf{C}^m))$  is the  $U_m$ -orbit through  $\operatorname{diag}(1, 1, 0, \dots, 0)$ . This proves that  $\tilde{\Psi}$  descends to a diffeomorphism  $\Psi: \mathbf{G}_2(\mathbf{C}^m) \xrightarrow{\cong} \mathcal{C}$ .

The complex vector space  $\mathcal{M}_{m \times 2}(\mathbf{C})$  is endowed with its classical Hermitian structure  $\langle A, B \rangle := \operatorname{tr}(AB^*)$ , with associated symplectic form  $\omega(\cdot, \cdot) = -\operatorname{Im} \langle \cdot, \cdot \rangle$ . The map  $\tilde{\Psi}$  above and the map  $\tilde{\Phi}: \mathcal{M}_{m \times 2}(\mathbf{C}) \longrightarrow \mathcal{H}_0(2)$  given by

$$\tilde{\Phi}(a, b) := (a, b)^* \cdot (a, b) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

are moment maps for the Hamiltonian actions of  $U_m$  and  $U_2$  respectively. One has  $\mathbf{V}_2(\mathbf{C}^m) = \tilde{\Phi}^{-1}(0)$  and thus  $\mathbf{G}_2(\mathbf{C}^m)$  occurs as symplectic reduction of the Hermitian vector space  $\mathcal{M}_{m \times 2}(\mathbf{C})$  and thereby inherits a  $U_m$ -invariant Kähler structure, using, for instance [Ki], §1.7. (Strictly speaking, one deals in [Ki] with compact Kähler manifolds; to fulfill this condition, one can first divide  $\mathcal{M}_{m \times 2}(\mathbf{C}) - \{0\}$  by the diagonal action of  $\mathbf{C}^*$  to put oneself into a complex projective space.) The residual map  $\Psi: \mathbf{G}_2(\mathbf{C}^m) \xrightarrow{\cong} \mathcal{C} \subset \mathcal{H}(m)$  is a moment map for the action of  $U_m$  on  $\mathbf{G}_2(\mathbf{C}^m)$ .

Being thus a Kähler manifold,  $\mathbf{G}_2(\mathbf{C}^m)$  is a Riemannian Poisson manifold. This structure descends to the complex orbifold  ${}^m\mathcal{P}^3$ : the algebra  $\mathcal{C}^\infty({}^m\mathcal{P}^3)$  admits a unique Lie bracket so that the projection  $\mathbf{G}_2(\mathbf{C}^m) \longrightarrow {}^m\mathcal{P}^3$  is a Poisson map.

(3.14) It is possible to endow with a Poisson structure the space  ${}^m\mathcal{P}\mathcal{P}_+^3$  of configurations of *all*  $m$ -gons in  $\mathbf{R}^3$ , without fixing the perimeter to 2. It suffices in the above construction, to replace the  $U_2$ -reduction  $\mathbf{G}_2(\mathbf{C}^m) = \tilde{\Phi}^{-1}(0)/U_2$  by the  $SU_2$ -reduction  $\tilde{\mathbf{G}}_2(\mathbf{C}^m) := \tilde{\Phi}^{-1}(0)/SU_2$ . The latter is a non-compact space, the total space of the determinant bundle over  $\mathbf{G}_2(\mathbf{C}^m)$  with the zero

section collapsed. The trace function on  $\mathcal{M}_{m \times 2}(\mathbf{C})$  descends to  $\tilde{\mathbf{G}}_2(\mathbf{C}^m)$  and to the Casimir function “perimeter” on  ${}^m\mathcal{P}\mathcal{P}_+^3$ .

4. POLYGONS WITH GIVEN SIDES – KÄHLER STRUCTURES

We now use the map  $\ell : {}^m\tilde{\mathcal{P}}^k, {}^m\mathcal{P}_+^k, {}^m\mathcal{P}^k \rightarrow \mathbf{R}^m$  defined in (2.4). Recall that  $\ell(\rho)$ , for  $\rho \in {}^m\tilde{\mathcal{P}}^k$ , is the length of the successive sides of a representative of  $r$  with total perimeter 2.

For  $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbf{R}_{\geq 0}^m$  with  $\sum_{i=1}^m \alpha_i = 2$ , we define

$${}^m\tilde{\mathcal{P}}^k(\alpha) :=: \tilde{\mathcal{P}}^k(\alpha) := \{\rho \in {}^m\tilde{\mathcal{P}}^k \mid \ell(\rho) = \alpha\} \subset {}^m\tilde{\mathcal{P}}^k.$$

The space  $\tilde{\mathcal{P}}^k(\alpha)$  is invariant under the action of  $O_k$ . We define the moduli spaces

$$\mathcal{P}_+^k(\alpha) := SO_k \backslash \tilde{\mathcal{P}}^k(\alpha) = \ell^{-1}(\alpha) \subset {}^m\mathcal{P}_+^k$$

and

$$\mathcal{P}^k(\alpha) := O_k \backslash \tilde{\mathcal{P}}^k(\alpha) = \ell^{-1}(\alpha) \subset {}^m\mathcal{P}^k.$$

The space  $\tilde{\mathcal{P}}^1(\alpha)$  consists of a finite number of points and is generically empty. We call  $\alpha$  *generic* if  $\tilde{\mathcal{P}}^1(\alpha) = \emptyset$ .

**THEOREM 4.1.** *The map  $\mu := \ell \circ \widehat{\Phi} : \mathbf{G}_2(\mathbf{C}^m) \rightarrow \mathbf{R}^m$  is a moment map for the action of  $U_1^m$  on  $\mathbf{G}_2(\mathbf{C}^m)$ .*

*Proof.* As seen in (3.13), the moment map  $\Psi : \mathbf{G}_2(\mathbf{C}^m) \rightarrow \mathcal{H}(m)$  for the  $U_m$ -action on  $\mathbf{G}_2(\mathbf{C}^m)$  is induced from  $\tilde{\Psi} : \mathcal{M}_{m \times 2}(\mathbf{C}) \rightarrow \mathcal{H}(m)$  given by  $\tilde{\Psi}(a, b) := (a, b) \cdot (a, b)^*$ . A moment map  $\mu$  for the action of  $U_1^m$  is obtained by composing  $\Psi$  with the projection  $\mathcal{H}(m) \rightarrow \mathbf{R}^m$  associating to a matrix its diagonal entries. So, if  $\Pi \in \mathbf{G}_2(\mathbf{C}^m)$  is generated by  $a$  and  $b$  with  $(a, b) \in \mathbf{V}_2(\mathbf{C}^m)$ , one has

$$\mu(\Pi) = (|a_1|^2 + |b_1|^2, \dots, |a_m|^2 + |b_m|^2) = \ell \circ \widehat{\Phi}(a, b). \quad \square$$

A now classic theorem of Atiyah and Guillemin-Sternberg [Au, §III.4.2] asserts that the image of a moment map for a torus action is a convex polytope (the *moment polytope*). The restriction of the moment map to the fixed point set of an anti-symplectic involution has the same image [Du]. In our case, one gets these facts directly :