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# AN ASYMPTOTIC FREIHEITSSATZ FOR FINITELY GENERATED GROUPS

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ABSTRACT. Given two fixed integers $k \geq 2$ and $l \geq 3$, let $\Gamma = \langle X \mid R \rangle$ be a presentation of the group $\Gamma$ with $k = \#X$ generators and $l = \#R$ relations. We show that the following property of presentations of groups is generic in the sense of Gromov: for any $y \in X$, the subgroup of $\Gamma$ generated by $X - \{y\}$ is free of rank $k-1$. This gives some generic estimates for the spectral radius of the adjacency operator in the Cayley graph of $\Gamma$ relative to the generating system $S = X \cup X^{-1}$.

## 1. INTRODUCTION

The existence of free subgroups in some finitely generated group $\Gamma$ gives some information about the structure of $\Gamma$. For example, it implies that $\Gamma$ is non-amenable, and in particular that $\Gamma$ has exponential growth. There are several results which ensure that various groups do have non-abelian free subgroups. For example:

THEOREM (Tits's alternative [15]). *Let $\Gamma$ be a finitely generated linear group. Then either $\Gamma$ is almost solvable or $\Gamma$ contains a free subgroup on two generators.*

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THEOREM (Magnus's Freiheitssatz [12]). *Let $\Gamma = \langle X \mid r \rangle$ be a one relator group, $x_0 \in X$ be a generator of $\Gamma$ that appears in the relation $r$ and $r$ be a cyclically reduced word in the free group $\mathbf{F}_X$ generated by $X$; then $X - \{x_0\}$ freely generates a free group in $\Gamma$.*

Our purpose in this work is to measure in some sense how frequent it is for a presentation $\Gamma = \langle X \mid R \rangle$ to be such that a proper subset of $X$ is free in $\Gamma$. We prove the following result:

THEOREM 1.1. *Let $\Gamma = \langle X \mid R \rangle$ be a finite presentation with $k$ generators, $l$ relations and any fixed $x_0$ in $X$. Then the fact that $X - \{x_0\}$ freely generates a free group in $\Gamma$ is generic in the sense of Gromov.*

The key idea is contained in proposition 4.1. Roughly speaking, if you choose at random $l$ long relations and if the presentation satisfies a Dehn algorithm, then every generator $x_0$ will appear in every sufficiently long subword of every relation and hence it will appear in every product of conjugates of relations. So $X - \{x_0\}$ generates a free group in $\Gamma$.

In [6], the first author has shown that "$X$ generates a free semi-group" is generic and that this implies bounds on the spectrum of the adjacency operator associated to the oriented Cayley graph of $\Gamma$ relative to $X$. In section 5 below, we consider the adjacency operator $h_S$ of the Cayley graph of $\Gamma$ relative to $S = X \cup X^{-1}$, and we prove similarly estimates on the norm of $h_S$.

After finishing this paper we discovered that a result similar to Theorem 1.1 has been proved, using different methods, by G. Arzhantseva and A. Ol′shanskii in [1]. They employed a slightly different definition of the genericity and they proved that the small cancellation condition $C'(\lambda)$ is generic with respect to this new definition.

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