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## 3. ABOUT GENERICITY

LEMMA 3.1. *Let  $X = \{x_1, \dots, x_k, y\}$ . For every  $0 < \epsilon < 1/(k+1)$ , the ratio*

$$\frac{\#\{r \in \mathbf{F}_X \mid |r| = n, r \text{ is } (\epsilon, y)\text{-balanced}\}}{\#\{r \in \mathbf{F}_X \mid |r| = n\}}$$

*tends to 1 when  $n$  tends  $\infty$ .*

*Proof.* First we want to rephrase the Lemma in terms of generating functions. Let  $K$  be any fixed subset of  $\mathbf{F}_X$  and  $F_K(z, u)$  be the generating function defined by

$$F_K(z, u) = \sum_{r \in K} z^{|r|} u^{n_y(r)}.$$

$F_K(z, u)$  strongly depends on the choice of the generator  $y$ . However, as  $y$  is fixed throughout the proof and to lighten the notation, we write  $F_K(z, u)$  instead of  $F_{y,K}(z, u)$ .

Defining  $c_{n,l}$  and  $p_n(l)$  by

$$F_{\mathbf{F}_X}(z, u) = \sum_{r \in \mathbf{F}_X} z^{|r|} u^{n_y(r)} = \sum_{n,l} c_{n,l} z^n u^l \quad \text{and} \quad p_n(l) = \frac{c_{n,l}}{\sum_m c_{n,m}},$$

we have to prove that for every  $0 < \epsilon < 1/(k+1)$ ,

$$\lim_{n \rightarrow \infty} \sum_{0 \leq l < \epsilon n} p_n(l) = 0.$$

We want to find an analytical form for  $F_{\mathbf{F}_X}(z, u)$ .

It is clear that if  $K_1$  and  $K_2$  are disjoint subsets of  $\mathbf{F}_X$  then  $F_{K_1 \cup K_2}(z, u) = F_{K_1}(z, u) + F_{K_2}(z, u)$ .

Let  $K_1, K_2$  be two subsets of  $\mathbf{F}_X$ ; assume that the map  $K_1 \times K_2 \rightarrow K_1 K_2$  defined by  $(\omega_1, \omega_2) \mapsto \omega_1 \omega_2$  is one to one and satisfies  $|\omega_1 \omega_2| = |\omega_1| + |\omega_2|$  for  $\omega_i \in K_i$  (where  $K_1 K_2 = \{\omega_1 \omega_2 \mid \omega_i \in K_i\}$ ); it is also clear that  $F_{K_1 K_2}(z, u) = F_{K_1}(z, u) F_{K_2}(z, u)$ . This can be extended to a finite product of such  $K_i$ 's.

First we compute the generating functions of some subsets  $K$  of  $\mathbf{F}_X$ .

- $F_{\{e\}}(z, u) = 1$ .
- Denote by  $X' = X - \{y\}$ . As there are exactly  $2k(2k-1)^{n-1}$  reduced words of length  $n \geq 1$  in  $\mathbf{F}_{X'}$ , we obtain  $F_{[\mathbf{F}_{X'} - \{e\}]}(z, u) = \frac{2kz}{1-(2k-1)z}$ . Set  $f(z, u) = F_{[\mathbf{F}_{X'} - \{e\}]}(z, u)$ .
- For  $\langle y \rangle = \{y^i \mid i \in \mathbf{Z} - \{0\}\}$ , we have  $F_{\langle y \rangle}(z, u) = \frac{2uz}{1-uz}$ , because there are exactly 2 elements  $y^{\pm i}$  in  $\langle y \rangle$  such that  $n_y(y^{\pm i}) = |y^{\pm i}| = i$ . Set  $h(z, u) = F_{\langle y \rangle}(z, u)$ .

Now we can partition  $\mathbf{F}_X$  as follows:

$$\mathbf{F}_X = \{e\} \amalg [\mathbf{F}_{X'} - \{e\}] \amalg_{n \geq 0} I_n$$

where

$$I_n = \{\omega_0 y^{i_1} \omega_1 \dots y^{i_{n-1}} \omega_{n-1} y^{i_n} \omega_n \mid \omega_j \in \mathbf{F}_{X'}, \omega_j \neq e \text{ for } j \neq 0 \text{ or } n, \text{ and } i_j \neq 0\}.$$

It is easy to check that  $F_{I_n}(z, u) = (f(z, u) + 1)^2 h(z, u) (h(z, u)f(z, u))^{n-1}$ . So we obtain that

$$\begin{aligned} F_{\mathbf{F}_X}(z, u) &= 1 + f(z, u) + \sum_{n \geq 1} (f(z, u) + 1)^2 h(z, u) (h(z, u)f(z, u))^{n-1} \\ &= (1 + f(z, u)) \left(1 + \frac{h(z, u)(f(z, u) + 1)}{1 - h(z, u)f(z, u)}\right) \\ &= \frac{(1 + z)(1 + uz)}{1 - (2k - 1)z - uz(1 + (2k + 1)z)}. \end{aligned}$$

Borrowing notation from [2], let  $g(z, u) = (1 + z)(1 + zu)$  and  $P(z, u) = 1 - (2k - 1)z - uz(1 + (2k + 1)z) = 1 - (2k - 1 + u)z - (2k + 1)uz^2$ . Then

$$F_{\mathbf{F}_X}(z, u) = \frac{g(z, u)}{P(z, u)}$$

and let  $r(s)$  be the root of smallest modulus of  $P(r(s), e^s) = 0$  in a small neighborhood of  $s = 0$ . In particular  $r(0) = \frac{1}{2k+1}$ . According to [2, (3.1)], we obtain from [2, Theorem 1] that

$$\lim_{n \rightarrow \infty} \sup_x \left| \sum_{k \leq \sigma_n x + \mu_n} p_n(k) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt \right| = 0$$

with  $\mu = \frac{r'(0)}{r(0)}$ ,  $\mu_n = n\mu = n \frac{r'(0)}{r(0)}$  and  $\sigma_n^2 = n\sigma^2 = n(\mu^2 - \frac{r''(0)}{r(0)})$ .

Computing  $r'(0)$  or easy combinatorial considerations gives  $\mu_n = \frac{n}{k+1}$ . The actual value of  $\sigma$  is here useless.

Now let  $\epsilon < \frac{1}{k+1}$  and  $\delta > 0$ . Let  $x$  such that  $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt < \delta$ . Then there exists  $N$  such that for  $n > N$ ,  $\epsilon n < \sigma \sqrt{nx} + \frac{n}{k+1}$  since  $\epsilon < \frac{1}{k+1}$ . Therefore, for  $n > N$ ,

$$\sum_{k < \epsilon n} p_n(k) \leq \sum_{k < \sigma_n x + \mu_n} p_n(k)$$

and there exists  $N_1$  such that for  $n > N_1$ ,

$$\sum_{k < \epsilon n} p_n(k) \leq 2\delta. \quad \square$$

COROLLARY 3.2. For  $\#X = k$ ,  $\#R = n$ ,  $x_0 \in X$  and  $0 < \epsilon < 1/k$  fixed, being  $(\epsilon, x_0)$ -balanced is generic for  $\Gamma = \langle X | R \rangle$ .

*Proof of corollary.* We choose  $n$  relations at random; by Lemma 3.1, every  $r \in R$  is generically  $(\epsilon, x_0)$ -balanced, but the conjunction of finitely many generic properties is also generic.  $\square$

#### 4. SOME SUFFICIENT CONDITIONS FOR THE EXISTENCE OF FREE SUBGROUPS

We first begin by a very easy proposition.

PROPOSITION 4.1. Let  $\Gamma = \langle X | R \rangle$  be a finite presentation, which has a Dehn algorithm and such that for some  $y \in X$  every subword  $u$  of every  $r \in R^*$  with  $|u| > |r|/2$  contains either  $y$  or  $y^{-1}$ , then  $X - \{y\}$  generates a free subgroup in  $\Gamma$ .

The proof of this proposition will follow from Lemma 4.2 below.

LEMMA 4.2. For  $\langle X | R \rangle$  a finite presentation of a group  $\Gamma$  and  $y \in X$ , the following are equivalent:

- $X - \{y\}$  freely generates a free subgroup of  $\Gamma$ ;
- every non trivial element  $\omega \in \mathbf{F}_X$ , which represents the identity in  $\Gamma$ , contains either  $y$  or  $y^{-1}$ .

*Proof.* 1)  $\Rightarrow$  2): By contraposition, suppose that there exists a non trivial reduced element  $\omega \in \mathbf{F}_{X-\{y\}}$  such that  $\bar{\omega} = e$  (where  $\bar{\omega}$  is the canonical projection of  $\omega$  in  $\Gamma$ ), then  $X - \{y\}$  does not freely generate a free subgroup in  $\Gamma$ .

2)  $\Rightarrow$  1): Let  $\omega_1, \omega_2 \in \mathbf{F}_{X-\{y\}}$  be two reduced elements such that  $\bar{\omega}_1 = \bar{\omega}_2 \in \Gamma$ . Then  $\overline{\omega_1 \omega_2^{-1}} = e \in \Gamma$ . So  $\omega_1 \omega_2^{-1}$  is an element of  $\mathbf{F}_{X-\{y\}}$  which represents the identity in  $\Gamma$ . By hypothesis, this implies  $\omega_1 = \omega_2$  in  $\mathbf{F}_X$ . Hence  $X - \{y\}$  freely generates a free subgroup in  $\Gamma$ .  $\square$

*Proof of Proposition 4.1.* By Lemma 4.2, it is sufficient to show that every non trivial reduced word on  $\mathbf{F}_X$  which represents the identity in  $\Gamma$  contains either  $y$  or  $y^{-1}$ . By assumption,  $\Gamma = \langle X | R \rangle$  satisfies a Dehn algorithm, so such a word contains at least one half of a relator  $r$  in  $R$  which contains at least one occurrence of  $y$  or  $y^{-1}$ .  $\square$