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Artikel: AN ASYMPTOTIC FREIHEITSSATZ FOR FINITELY GENERATED GROUPS

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3. About genericity

LEMMA 3.1. *Let* $X = \{x_1, \dots, x_k, y\}$. *For every* $0 < \epsilon < 1/(k+1)$, *the ratio*

$$\frac{\#\{r \in \mathbf{F}_X \mid |r| = n, r \text{ is } (\epsilon, y)\text{-balanced}\}}{\#\{r \in \mathbf{F}_X \mid |r| = n\}}$$

tends to 1 *when* n *tends* ∞.

Proof. First we want to rephrase the Lemma in terms of generating functions. Let K be any fixed subset of $\mathbf{F}_X$ and $F_K(z, u)$ be the generating function defined by

$$F_K(z, u) = \sum_{r \in K} z^{|r|} u^{n_y(r)} .$$

$F_K(z, u)$ strongly depends on the choice of the generator y. However, as y is fixed throughout the proof and to lighten the notation, we write $F_K(z, u)$ instead of $F_{y,K}(z, u)$.

Defining $c_{n,l}$ and $p_n(l)$ by

$$F_{\mathbf{F}_X}(z, u) = \sum_{r \in \mathbf{F}_X} z^{|r|} u^{n_y(r)} = \sum_{n,l} c_{n,l} z^n u^l \quad \text{and} \quad p_n(l) = \frac{c_{n,l}}{\sum_m c_{n,m}} ,$$

we have to prove that for every $0 < \epsilon < 1/(k+1)$,

$$\lim_{n \to \infty} \sum_{0 \le l < \epsilon n} p_n(l) = 0 .$$

We want to find an analytical form for $F_{\mathbf{F}_X}(z, u)$.

It is clear that if K_1 and K_2 are disjoint subsets of $\mathbf{F}_X$ then $F_{K_1 \cup K_2}(z, u) = F_{K_1}(z, u) + F_{K_2}(z, u)$.

Let K_1, K_2 be two subsets of F_X; assume that the map $K_1 \times K_2 \to K_1 K_2$ defined by $(\omega_1, \omega_2) \mapsto \omega_1 \omega_2$ is one to one and satisfies $|\omega_1 \omega_2| = |\omega_1| + |\omega_2|$ for $\omega_i \in K_i$ (where $K_1 K_2 = \{\omega_1 \omega_2 | \omega_i \in K_i\}$); it is also clear that $F_{K_1 K_2}(z, u) = F_{K_1}(z, u) F_{K_2}(z, u)$. This can be extended to a finite product of such K_i's.

First we compute the generating functions of some subsets K of $\mathbf{F}_X$.

- $F_{\{e\}}(z, u) = 1$.
- Denote by $X' = X - \{y\}$. As there are exactly $2k(2k-1)^{n-1}$ reduced words of length $n \ge 1$ in $\mathbf{F}_{X'}$, we obtain $F_{[\mathbf{F}_{X'} - \{e\}]}(z, u) = \frac{2kz}{1-(2k-1)z}$. Set $f(z, u) = F_{[\mathbf{F}_{X'} - \{e\}]}(z, u)$.
- For $\langle y \rangle = \{y^i \mid i \in \mathbf{Z} - \{0\}\}$, we have $F_{\langle y \rangle}(z, u) = \frac{2uz}{1-uz}$, because there are exactly 2 elements $y^{\pm i}$ in $\langle y \rangle$ such that $n_y(y^{\pm i}) = |y^{\pm i}| = i$. Set $h(z, u) = F_{\langle y \rangle}(z, u)$.

Now we can partition $\mathbf{F}_X$ as follows:

$$\mathbf{F}_X = \{e\} \amalg [\mathbf{F}_{X'} - \{e\}] \coprod_{n\geq 0} I_n$$

where

$$I_n = \{\omega_0 y^{i_1} \omega_1 \dots y^{i_{n-1}} \omega_{n-1} y^{i_n} \omega_n \mid \omega_j \in \mathbf{F}_{X'},\ \omega_j \neq e \text{ for } j \neq 0 \text{ or } n, \text{ and } i_j \neq 0\}.$$

It is easy to check that $F_{I_n}(z, u) = (f(z, u) + 1)^2 h(z, u) (h(z, u)f(z, u))^{n-1}$. So we obtain that

$$\begin{aligned} F_{\mathbf{F}_X}(z, u) &= 1 + f(z, u) + \sum_{n\geq 1} (f(z, u) + 1)^2 h(z, u) (h(z, u) f(z, u))^{n-1} \\ &= (1 + f(z, u))\left(1 + \frac{h(z, u)(f(z, u) + 1)}{1 - h(z, u)f(z, u)}\right) \\ &= \frac{(1 + z)(1 + uz)}{1 - (2k - 1)z - uz(1 + (2k + 1)z)}. \end{aligned}$$

Borrowing notation from [2], let $g(z, u) = (1 + z)(1 + zu)$ and $P(z, u) = 1 - (2k - 1)z - uz(1 + (2k + 1)z) = 1 - (2k - 1 + u)z - (2k + 1)uz^2$. Then

$$F_{\mathbf{F}_X}(z, u) = \frac{g(z, u)}{P(z, u)}$$

and let $r(s)$ be the root of smallest modulus of $P(r(s), e^s) = 0$ in a small neighborhood of $s = 0$. In particular $r(0) = \frac{1}{2k+1}$. According to [2, (3.1)], we obtain from [2, Theorem 1] that

$$\lim_{n\to\infty} \sup_x \left| \sum_{k\leq \sigma_n x + \mu_n} p_n(k) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \right| = 0$$

with $\mu = \frac{r'(0)}{r(0)}$, $\mu_n = n\mu = n\frac{r'(0)}{r(0)}$ and $\sigma_n^2 = n\sigma^2 = n(\mu^2 - \frac{r''(0)}{r(0)})$.

Computing $r'(0)$ or easy combinatorial considerations gives $\mu_n = \frac{n}{k+1}$. The actual value of σ is here useless.

Now let $\epsilon < \frac{1}{k+1}$ and $\delta > 0$. Let x such that $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt < \delta$. Then there exists N such that for $n > N$, $\epsilon n < \sigma\sqrt{n}x + \frac{n}{k+1}$ since $\epsilon < \frac{1}{k+1}$. Therefore, for $n > N$,

$$\sum_{k<\epsilon n} p_n(k) \leq \sum_{k<\sigma_n x + \mu_n} p_n(k)$$

and there exists N_1 such that for $n > N_1$,

$$\sum_{k<\epsilon n} p_n(k) \leq 2\delta. \quad \square$$

COROLLARY 3.2. *For* $\#X = k$, $\#R = n$, $x_0 \in X$ *and* $0 < \epsilon < 1/k$ *fixed, being* (ϵ, x_0)*-balanced is generic for* $\Gamma = \langle X \mid R \rangle$.

Proof of corollary. We choose n relations at random; by Lemma 3.1, every $r \in R$ is generically (ϵ, x_0)-balanced, but the conjunction of finitely many generic properties is also generic. □

4. SOME SUFFICIENT CONDITIONS FOR THE EXISTENCE OF FREE SUBGROUPS

We first begin by a very easy proposition.

PROPOSITION 4.1. *Let* $\Gamma = \langle X \mid R \rangle$ *be a finite presentation, which has a Dehn algorithm and such that for some* $y \in X$ *every subword* u *of every* $r \in R^*$ *with* $|u| > |r|/2$ *contains either* y *or* y^{-1}, *then* $X - \{y\}$ *generates a free subgroup in* Γ.

The proof of this proposition will follow from Lemma 4.2 below.

LEMMA 4.2. *For* $\langle X \mid R \rangle$ *a finite presentation of a group* Γ *and* $y \in X$, *the following are equivalent:*

- $X - \{y\}$ *freely generates a free subgroup of* Γ;
- *every non trivial element* $\omega \in \mathbf{F}_X$, *which represents the identity in* Γ, *contains either* y *or* y^{-1}.

Proof. 1) $\Rightarrow$ 2) : By contraposition, suppose that there exists a non trivial reduced element $\omega \in \mathbf{F}_{X-\{y\}}$ such that $\overline{\omega} = e$ (where $\overline{\omega}$ is the canonical projection of ω in Γ), then $X - \{y\}$ does not freely generate a free subgroup in Γ.

2) $\Rightarrow$ 1) : Let $\omega_1, \omega_2 \in \mathbf{F}_{X-\{y\}}$ be two reduced elements such that $\overline{\omega}_1 = \overline{\omega}_2 \in \Gamma$. Then $\overline{\omega_1\omega_2^{-1}} = e \in \Gamma$. So $\omega_1\omega_2^{-1}$ is an element of $\mathbf{F}_{X-\{y\}}$ which represents the identity in Γ. By hypothesis, this implies $\omega_1 = \omega_2$ in $\mathbf{F}_X$. Hence $X - \{y\}$ freely generates a free subgroup in Γ. □

Proof of Proposition 4.1. By Lemma 4.2, it is sufficient to show that every non trivial reduced word on $\mathbf{F}_X$ which represents the identity in Γ contains either y or y^{-1}. By assumption, $\Gamma = \langle X \mid R \rangle$ satisfies a Dehn algorithm, so such a word contains at least one half of a relator r in R which contains at least one occurrence of y or y^{-1}. □