4. SOME SUFFICIENT CONDITIONS FOR THE EXISTENCE OF FREE SUBGROUPS

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **44 (1998)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **13.09.2024**

COROLLARY 3.2. *For* $\#X = k$, $\#R = n$, $x_0 \in X$ *and* $0 < \epsilon < 1/k$ *fixed, being* (ϵ, x_0)*-balanced is generic for* $\Gamma = \langle X | R \rangle$.

Proof of corollary. We choose n relations at random; by Lemma 3.1, every $r \in R$ is generically (ϵ, x_0)-balanced, but the conjunction of finitely many generic properties is also generic. □

4. SOME SUFFICIENT CONDITIONS FOR THE EXISTENCE OF FREE SUBGROUPS

We first begin by a very easy proposition.

PROPOSITION 4.1. *Let* $\Gamma = \langle X \mid R \rangle$ *be a finite presentation, which has a Dehn algorithm and such that for some* $y \in X$ *every subword* u *of every* $r \in R^*$ *with* $|u| > |r|/2$ *contains either* y *or* y^{-1}, *then* $X - \{y\}$ *generates a free subgroup in* Γ.

The proof of this proposition will follow from Lemma 4.2 below.

LEMMA 4.2. *For* $\langle X | R \rangle$ *a finite presentation of a group* Γ *and* $y \in X$, *the following are equivalent:*

- $X - \{y\}$ *freely generates a free subgroup of* Γ;
- *every non trivial element* $\omega \in \mathbf{F}_X$, *which represents the identity in* Γ, *contains either* y *or* y^{-1}.

Proof. 1) $\Rightarrow$ 2) : By contraposition, suppose that there exists a non trivial reduced element $\omega \in \mathbf{F}_{X-\{y\}}$ such that $\overline{\omega} = e$ (where $\overline{\omega}$ is the canonical projection of ω in Γ), then $X - \{y\}$ does not freely generate a free subgroup in Γ.

2) $\Rightarrow$ 1) : Let $\omega_1, \omega_2 \in \mathbf{F}_{X-\{y\}}$ be two reduced elements such that $\overline{\omega}_1 = \overline{\omega}_2 \in \Gamma$. Then $\overline{\omega_1 \omega_2^{-1}} = e \in \Gamma$. So $\omega_1 \omega_2^{-1}$ is an element of $\mathbf{F}_{X-\{y\}}$ which represents the identity in Γ. By hypothesis, this implies $\omega_1 = \omega_2$ in $\mathbf{F}_X$. Hence $X - \{y\}$ freely generates a free subgroup in Γ. □

Proof of Proposition 4.1. By Lemma 4.2, it is sufficient to show that every non trivial reduced word on $\mathbf{F}_X$ which represents the identity in Γ contains either y or y^{-1}. By assumption, $\Gamma = \langle X | R \rangle$ satisfies a Dehn algorithm, so such a word contains at least one half of a relator r in R which contains at least one occurrence of y or y^{-1}. □

The interest of this proposition appears when we replace "having a Dehn's algorithm" by "satisfying the small cancellation condition $C'(1/6)$", because $C'(1/6)$ and the fact that every subword u of any relation r with $|u| > |r|/2$ contains at least one y or y^{-1} are easy to check on a given presentation.

Unfortunately, as explained before, it is not known if the small cancellation hypothesis is generic, so we need other sufficient conditions to ensure that $X - \{y\}$ generates a free subgroup in Γ.

PROPOSITION 4.3. *Let $\Gamma = \langle X \mid R\rangle$ be a finite presentation with k generators and l relations, which is (ϵ, x_0)-balanced for some $0 < \epsilon < 1/k$ and some $x_0 \in X$, and which satisfies a θ-condition such that $\theta < \epsilon/(2-\epsilon)$. Then $X - \{x_0\}$ freely generates a free group in Γ.*

To prove the proposition we need the following lemma and the following notations. For a cell f_i of the diagram, we denote by *Int*(f_i) (resp. by *Ext*(f_i)) the number of edges of f_i which are internal to the diagram (resp. which are on the border of the diagram). We denote also by #(f_i) the total number of edges of the cell f_i.

LEMMA 4.4. *Let $\Gamma = \langle X|R\rangle$ be a finite presentation of a group Γ which satisfies a θ-condition for some $0 < \theta < 1$, then for every reduced diagram, there exists a 2-cell f of Δ satisfying*

$$Int(f) \leq \frac{2\theta}{1+\theta} \#(f)\,.$$

Proof. First we prove it for simple diagrams. Let $\epsilon = 2\theta/(1+\theta)$. Because the diagram is simple we have the following equalities:

I) $\sum_i Ext(f_i) = E(\Delta) = |\partial\Delta|$,

II) $\sum_i Int(f_i) = 2I(\Delta)$, because every internal edge belongs to two different cells.

So we get:

$$\#(\Delta) = \frac{1}{2}\sum_i Int(f_i) + \sum_i Ext(f_i) = \sum_i \#(f_i) - \frac{1}{2}\sum_i Int(f_i)\,.$$

To obtain a contradiction, we suppose that every cell f_i of one diagram Δ is such that $(1/\epsilon)Int(f_i) > \#(f_i)$. Then we have

$$\frac{1}{\epsilon}\sum_i Int(f_i) > \sum_i \#(f_i) = \#(\Delta) + \frac{1}{2}\sum_i Int(f_i),$$

whence $\frac{2-\epsilon}{2\epsilon}\sum_i Int(f_i) > \#(\Delta)$ or $\frac{2-\epsilon}{\epsilon}I(\Delta) > \#(\Delta)$. Since $\epsilon = 2\theta/(1+\theta)$, we obtain $I(\Delta) > \theta\#(\Delta)$, which contradicts the θ-condition.

In fact, if the reduced diagram Δ is not simple, it is a union of simple diagrams linked by bridges. So each of its parts, which is a simple diagram, defines another reduced diagram (relative to another word), so the inequality holds for every part of Δ which is a simple diagram. We conclude by saying that increasing the number of external edges does not affect the inequality. □

Proof of 4.3. By Lemma 4.2, it is sufficient to prove that the (ϵ, x_0)-balanced and θ-conditions imply that every non trivial reduced word in $\mathbf{F}_X$ which vanishes in Γ contains at least one $x_0^{\pm 1}$.

Let us choose such a word ω and Δ a reduced diagram of ω. By Lemma 4.4, there exists a cell f with border equal to one $r \in R^*$, such that

$$Int(f) \leq \frac{2\theta}{1+\theta}\#(f) = \frac{2\theta}{1+\theta}|r| < \epsilon|r| \leq n_{x_0}(r),$$

because $\theta < \epsilon/(2-\epsilon)$. As there are more occurences of x_0 or x_0^{-1} than the number of internal edges, it means that some occurrences of x_0 or x_0^{-1} will be external edges, i.e. will be in the border of Δ which is ω. □

We are now able to prove the main theorem.

Proof of theorem 1.1. By Proposition 4.3, for a finite presentation $\langle X \mid R \rangle$, we know that being (ϵ, x_0)-balanced and satisfying a θ-condition is sufficient to ensure that $X - \{x_0\}$ freely generates a free subgroup in Γ. But by Corollary 3.2 and [13, Theorem 2], these two conditions are generic and so is the conjunction of these two conditions. □

5. Spectral estimates for adjacency operators on Cayley graphs

The existence of a free subgroup generated by $X - \{x_0\}$ gives an upper bound for the spectral value of the adjacency operator on the Cayley graph of $\Gamma = \langle X \mid R \rangle$ associated with the symmetric generating system $S = X \cup X^{-1}$.

We briefly recall some definitions and notations. The Cayley graph $G(\Gamma, X)$ of Γ associated with S has its set of vertices in bijection with Γ and two