§2. CATEGORIES, ALGEBRAS AND CELL REPRESENTATIONS

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§2. CATEGORIES, ALGEBRAS AND CELL REPRESENTATIONS

In this section we shall define the affine Temperley-Lieb algebras as the sets of endomorphisms in a category $\mathbf{T}^a$ (the "affine Temperley-Lieb category") which is an enrichment of $\mathbf{D}^a$, the category of affine diagrams defined in the last section. We shall construct an uncountable set of representations for these algebras by defining functors from this category to the category of modules over a ring R. It will turn out that these functors provide a "complete set" of representations for the affine Temperley-Lieb algebras. As in the case of diagrams, we shall begin with the finite case.

(2.1) DEFINITION. Let R be a (commutative, associative, unital) ring with an invertible element q. Write $\delta = -(q+q^{-1})$. The *Temperley-Lieb category* $\mathbf{T} = \mathbf{T}_{R,q}$ is defined as follows.

(1) The objects are the non-negative integers.

(2) If $t, n \in \mathbf{Z}_{\geq 0}$, the morphism set $\mathbf{T}(t,n)$ is the free R-module spanned by finite diagrams $: t \to n$.

(3) The composition of finite diagrams $\alpha\colon t \to n$ and $\beta\colon s \to t$ is $\alpha\beta := \delta^{m(\alpha,\beta)}\alpha \circ \beta = (-q-q^{-1})^{m(\alpha,\beta)}\alpha\circ\beta$. Extend bilinearly to define composition in $\mathbf{T}$.

Since q and R will generally be determined by the context, we shall usually suppress them.

The *Temperley-Lieb algebra* $\mathbf{T}(n) = \mathbf{T}(n,n)$ is cellular in the sense of [GL]. It follows that it possesses a family of "cell representations" with canonical bilinear forms. When R is a field, the heads of this family of modules form a complete set of irreducibles for the algebra. Suppose W is any functor $: \mathbf{T} \to R$-**mod** from $\mathbf{T}$ to the category R-**mod** of R-modules. Then for $n \in \mathbf{Z}_{\geq 0}$, $W(n)$ is clearly a $\mathbf{T}(n)$-module, so that W provides representations of all the Temperley-Lieb algebras simultaneously. Such functors will therefore be referred to as representations of the category $\mathbf{T}$, or $\mathbf{T}$-modules (see (2.3) below). We show next how the cell modules may be constructed from representations of the category.

(2.2) DEFINITION. Let t be a non-negative integer. The *cell representation* W_t of $\mathbf{T}$ is defined as follows.

(1) For $n \in \mathbf{Z}_{\geq 0}$, $W_t(n)$ is the free R-submodule generated by monic finite diagrams $\mu\colon t \to n$.

(2) If $s, n \in \mathbf{Z}_{\geq 0}$ and $\alpha: s \to n$ is a finite diagram, define $W_t(\alpha): W_t(s) \to W_t(n)$ by stipulating that for any finite monic diagram $\mu: t \to s$, $W_t(\alpha)(\mu) = \alpha * \mu$, where

$$\alpha * \mu = \begin{cases} \alpha\mu & \text{if } \alpha \circ \mu \text{ is monic,} \\ 0 & \text{otherwise.} \end{cases}$$

Extend this definition using linearity to obtain the required R-module homomorphism $W_t(\alpha)$.

(3) Let $\langle\ ,\ \rangle_t$ denote the R-bilinear form $W_t(n) \times W_t(n) \to R$ which takes monomorphisms $\mu, \nu: t \to n$ to

$$\langle \mu, \nu \rangle_t = \begin{cases} (-q - q^{-1})^{m(\nu^*, \mu)} & \text{if } \nu^* \circ \mu \text{ is monic,} \\ 0 & \text{otherwise.} \end{cases}$$

(2.3) DEFINITION. A $\mathbf{T}$-module is a functor from the Temperley-Lieb category to the category of R-modules. Parts (1) and (2) of (2.2) define $\mathbf{T}$-modules W_t (for $t \in \mathbf{Z}_{\geq 0}$). The form defined in (3) above is *invariant* in the sense that

$$\langle \alpha * \mu, \nu \rangle_t = \langle \mu, \alpha^* * \nu \rangle_t$$

and so we obtain further $\mathbf{T}$-modules rad_t and L_t where $\mathrm{rad}_t(n)$ is the *radical*

$$\{x \in W_t(n) \mid \langle x, y \rangle_t = 0 \text{ if } y \in W_t(n)\}$$

of this bilinear form and $L_t(n) = W_t(n)/\mathrm{rad}_t(n)$. For $n \in \mathbf{Z}_{\geq 0}$, let $\Lambda(n) = \{t \in \mathbf{Z}_{\geq 0} \mid t \leq n,\ t \equiv n \mod 2\}$ except if $q + q^{-1} = 0$ and n is nonzero and even, in which case we exclude 0 from $\Lambda(n)$. The set $\Lambda(n)$ parametrises the nonzero quotients $L_t(n)$.

(2.4) THEOREM [GL 2.6, 3.2, 3.4]. *Let R be a field and suppose $q \in R$ is nonzero. Let $n \in \mathbf{Z}_{\geq 0}$.*

(1) *If $t \in \mathbf{Z}_{\geq 0}$, M is a $\mathbf{T}(n)$-submodule of the cell module $W_t(n)$ and $s \in \Lambda(n)$ (2.3) is such that there exists a nonzero $\mathbf{T}(n)$-homomorphism $f: W_s(n) \to W_t(n)/M$, then $s \geq t$. If $s = t$, then $f(x) = rx + M$ for some nonzero element r in R.*

(2) *If $s \in \Lambda(n)$, then the radical of $W_s(n)$ as $\mathbf{T}(n)$-module is $\mathrm{rad}_s(n)$, the radical of the form $\langle \mu, \nu \rangle_t$.*

(3) *The family $L_s(n)$ indexed by $s \in \Lambda(n)$ is a complete set of irreducible $\mathbf{T}(n)$-modules.*

We shall now proceed with the affine analogue of (2.4).

(2.5) DEFINITION. Let R be a (commutative, unital) ring with an invertible element q. The *affine Temperley-Lieb category* $\mathbf{T}^a = \mathbf{T}^a_{R,q}$ is defined as follows.

(1) The objects are the non-negative integers.

(2) The morphism set $\mathbf{T}^a(t,n) = \mathbf{T}^a_{R,q}(t,n)$ is the free R-module spanned by the affine diagrams $: t \to n$.

(3) Composition is defined as the R-bilinear map which takes diagrams $\alpha\colon t \to n$ and $\beta\colon s \to t$ to the product $\alpha\beta := \delta^{m(\alpha,\beta)}\alpha \circ \beta = (-q-q^{-1})^{m(\alpha,\beta)}\alpha \circ \beta$, where $m(\alpha,\beta)$ is defined in (1.4).

As in the case of the Temperley-Lieb category, we shall generally omit the subscript (R,q).

We leave it to the reader to check that composition is associative (cf. (1.5)(2)) and that the above definition therefore does make $\mathbf{T}^a$ into a category.

We next define the set which will index the representations of the category $\mathbf{T}^a$ which we shall construct below.

(2.6) DEFINITION. Let Λ^a be the quotient of the set of pairs (t,z) where t is a non-negative integer and z is an invertible element of R by the relation which identifies $(0,z)$ with $(0,z^{-1})$ for all nonzero $z \in R$. Fix $(t,z) \in \Lambda^a$ and define $\chi = \chi_{t,z}\colon \mathbf{T}^a(t,t) \to R$ as the unique R-algebra homomorphism which annihilates non-monic diagrams and is given elsewhere by

$$\begin{aligned} \tau_0{}^i &\mapsto (z+z^{-1})^i && \text{if } t=0, \\ \tau_t^i &\mapsto z^i && \text{if } t>0. \end{aligned}$$

The (affine) *cell representation* $W_{t,z}$ is the functor from $\mathbf{T}^a$ to R-**mod** defined as follows.

(1) If n is a non-negative integer, $W_{t,z}(n)$ is the R-module generated by monic (affine) diagrams $\mu\colon t \to n$ subject to the relation:

$$\mu \circ \sigma = \chi_{t,z}(\sigma)\mu \qquad \text{if } \sigma\colon t \to t \text{ is monic.}$$

(2) There is an obvious R-bilinear action $\mathbf{T}^a(s,n) \times W_{t,z}(s) \to W_{t,z}(n)$ which takes a diagram $\alpha\colon s \to n$ and monic diagram $\mu\colon t \to s$ to

(2.6.1) $$\alpha * \mu = \begin{cases} \alpha\mu & \text{if } \alpha \circ \mu \text{ is monic,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbf{T}^a(\alpha)$ is the R-module homomorphism defined by $\mu \mapsto \alpha * \mu$.

The bilinear forms of the finite case are replaced by pairings between related couples of cell modules, which we now define. Let $\langle\ ,\ \rangle_{t,z}$ denote the R-bilinear pairing $W_{t,z}(n) \times W_{t,z^{-1}}(n) \longrightarrow R$ which takes monic diagrams $\mu, \nu \colon t \longrightarrow n$ to

$$\langle \mu, \nu \rangle_{t,z} = \begin{cases} \chi_{t,z}(\nu^* \mu) & \text{if } \nu^* \circ \mu \text{ is monic,} \\ 0 & \text{otherwise.} \end{cases}$$

(2.7) REMARKS. The R-module $W_{t,z}(n)$ is a module for the *affine Temperley-Lieb algebra* $\mathbf{T}^a(n) = \mathbf{T}^a(n,n)$. It has a basis of standard diagrams $: t \longrightarrow n$, because every monic diagram factors uniquely through its image by (1.7.1).

The pairing defined by (2.6.1) is invariant under the $\mathbf{T}^a(n)$ action (see (2.3) for the meaning of invariance). Hence we obtain $\mathbf{T}^a$-modules $\operatorname{rad}_{t,z}$ and $L_{t,z}$ where $\operatorname{rad}_{t,z}(n)$ is the radical in $W_{t,z}(n)$ of this pairing (i.e. the annihilator of $W_{t,z^{-1}}(n)$) and $L_{t,z}(n)$ is $W_{t,z}(n)/\operatorname{rad}_{t,z}(n)$.

For $n \in \mathbf{Z}_{\geq 0}$, let $\Lambda^a(n) = \{(t,z) \in \Lambda^a \mid t \leq n,\ t \equiv n \mod 2\}$, with the pair $(0,q)$ ($\equiv (0,q^{-1})$) removed if $q^2 = -1$ and n is nonzero and even. This set parametrises the nonzero $\mathbf{T}^a(n)$-modules $L_{t,z}(n)$. To see this, we have only to show that $\langle\ ,\ \rangle_{t,z} \neq 0$ for $(t,z) \in \Lambda^a(n)$. Write $k = (n-t)/2$ and denote by η^k the standard diagram $\eta_n \eta_{n-2} \dots \eta_{t+4} \eta_{t+2} \colon t \longrightarrow n$. One then verifies easily that

(2.7.1) $$\langle \eta^k, \eta^k \rangle = (-q - q^{-1})^k,$$

(2.7.2) $$\langle \tau_n \eta^k, \eta^k \rangle = \begin{cases} \chi(\tau_k) = z & \text{if } t > 0, \\ \chi(\tau_0) = z + z^{-1} & \text{if } t = 0, \end{cases}$$

whence the bilinear pairing $\langle\ ,\ \rangle_{t,z}$ is nonzero unless $q^2 = -1$ and $(t,z) = (0,q)$.

(2.8) THEOREM. *Let R be a field with $q \in R$ a nonzero element. Let n be a non-negative integer and $\mathbf{T}^a(n)$ be the affine Temperley-Lieb algebra (2.7).*

(1) *Let $(t,z) \in \Lambda^a$ (see (2.7)), let N be a $\mathbf{T}^a(n)$-submodule of the cell module $W_{t,z}(n)$ (2.6) and take $(s,y) \in \Lambda^a(n)$. Suppose that $f \colon W_{s,y}(n) \longrightarrow W_{t,z}(n)/N$ is a nonzero $\mathbf{T}^a(n)$-homomorphism. Then $s \geq t$. If $s = t$, then $(s,y) = (t,z)$ and $f(x) = rx + N$ for some r in R.*

(2) *For any $(s,y) \in \Lambda^a(n)$, the radical of $W_{s,y}(n)$ as a $\mathbf{T}^a(n)$-module is* $\operatorname{rad}_{s,y}(n)$.

(3) *If R is algebraically closed, then the family $L_{s,y}(n)$ indexed by $(s,y) \in \Lambda^a(n)$ is a complete set of distinct irreducible $\mathbf{T}^a(n)$-modules.*

Proof. The proofs of (1) and (2) are the same as those of [GL 2.6, 3.2, 3.4], given (1.7.2) and recalling that the bilinear forms $\phi_{t,z}$ are non-zero on the modules under consideration. From (1) and (2) it follows that $L_{s,y}(n)$ is an (absolutely) irreducible $\mathbf{T}^a(n)$-module for any $(s,y) \in \Lambda^a(n)$ and that these modules are pairwise inequivalent. Let M be an arbitrary finite dimensional irreducible $\mathbf{T}^a(n)$-module; assuming that $M \neq 0$ (as we may), we shall show that $M \cong L_{t,z}$ for some $(t,z) \in \Lambda^a$. Let $t \in \mathbf{Z}_{\geq 0}$ be minimal such that $\alpha . m \neq 0$ ($.$ denoting the module action) for some $m \in M$ and $\alpha : n \to n$ with t through strings (i.e. $t(\alpha) = t$). Since $M \neq 0$ such t, α and m exist; fix them for the rest of this proof. We shall find an invertible element z in R and construct a nonzero homomorphism $\theta : W_{t,z}(n) \to M$. If $q^2 = -1$ and $(t,z) = (0,q)$ for this θ, then α annihilates $W_{t,z}(n)$ when $t(\alpha) = 0$, contradicting our choice of t. Hence if such a θ exists, $(t,z) \in \Lambda^a(n)$. Moreover since M is semisimple, θ factors through its maximal semisimple quotient, which is $L_{t,z}(n)$ by parts (1) and (2). Hence to complete the proof of (3), it suffices to construct the homomorphism θ as above.

Let $\widetilde{W}_t(n)$ denote the free R-module on the set of monic diagrams $\mu : t \to n$. There is a $\mathbf{T}^a(n)$-action on $\widetilde{W}_t(n)$ given by

$$\alpha * \mu = \begin{cases} \alpha\mu & \text{if } \alpha \circ \mu \text{ is monic,} \\ 0 & \text{otherwise.} \end{cases}$$

Now $\alpha = i(\alpha) \circ \rho$ for a unique diagram $\rho : n \to t$ with ρ^* monic (cf.(1.7.1)). We therefore have a homomorphism $f : \widetilde{W}_t(n) \to M$ of $\mathbf{T}^a(n)$-modules given by $f(\mu) = \psi(\mu) . m$, where $\psi(\mu) = \mu\rho$, with ρ as above. This map is nonzero since $f(i(\alpha)) \neq 0$. Hence f is surjective and there is an element $y \in \widetilde{W}_t(n)$ such that $\psi(y) . m = m$.

Let $\nu : t \to n$ and $\sigma : t \to t$ be a pair of monic diagrams. We shall show that

$$\psi(\nu)\psi(y\sigma) . m = \psi(\nu\sigma) . m . \tag{2.8.1}$$

To see this, observe first that any element x of $\widetilde{W}_t(n)$ may be written uniquely in the form $x = \sum_\mu \mu x_\mu$, where the sum is over the standard diagrams $\mu : t \to n$ and the x_μ are in $\widetilde{W}_t(t)$. Write $y = \sum_\mu \mu y_\mu$ accordingly and note that since $\widetilde{W}_t(t)$ is an abelian algebra, if $\rho \circ \mu$ is monic, then $\rho\mu$, y_μ and σ commute with each other. It follows that

$$\begin{aligned} \psi(\nu)\psi(y\sigma) . m &= \sum_\mu (\nu\rho)(\mu y_\mu \sigma\rho) . m = \sum_\mu (\nu\sigma)\rho\mu y_\mu \rho . m \\ &= \psi(\nu\sigma)\psi(y) . m = \psi(\nu\sigma) . m, \end{aligned}$$

which proves (2.8.1).

Let σ_1 and σ_2 be monic diagrams from t to t. Taking $\nu = y\sigma_1$ and replacing σ by σ_2 in (2.8.1), we have

(2.8.2) $$\psi(y\sigma_1)\psi(y\sigma_2)\,.\,m = \psi(y\sigma_1\sigma_2)\,.\,m\,.$$

Hence $\widetilde{W}_t(t)$ has an action on the subspace V of M consisting of $\{\psi(y\sigma)\,.\,m \mid \sigma \in \widetilde{W}_t(t)\}$, the element $\sigma \in \widetilde{W}_t(t)$ acting via $\psi(y\sigma)$. Since R is algebraically closed, $\psi(y\tau_t)$ has a nonzero eigenvector $m' = \psi(y\sigma')\,.\,m \in V$, with corresponding eigenvalue ζ (say). Now take $(t, z) \in \Lambda^a$ such that t is as above, $z = \zeta$ if $t > 0$ or $\zeta = z + z^{-1}$ if $t = 0$. Define the character $\chi\colon \widetilde{W}_t(t) \to R$ as in (2.6). Then it follows from (2.8.2) that for any element $\sigma \in \widetilde{W}_t(t)$, we have

$$\psi(y\sigma)\,.\,m' = \chi(\sigma)m'\,.$$

Moreover for any monic diagram $\mu\colon t \to n$, we have

$$\begin{aligned}\psi(\mu\sigma)\,.\,m' &= \psi(\mu\sigma)\psi(y\sigma')\,.\,m = \psi(\mu\sigma\sigma')\,.\,m\\ &= \psi(\mu)\psi(y\sigma\sigma')\,.\,m = \psi(\mu)\psi(y\sigma)\psi(y\sigma')\,.\,m\\ &= \psi(\mu)\psi(y\sigma)\,.\,m'\\ &= \chi(\sigma)\psi(\mu)\,.\,m'\,.\end{aligned}$$

It follows that there is a nonzero homomorphism $\theta\colon W_{t,z}(n) \to M$ of $\mathbf{T}^a(n)$-algebras given by $\theta(\mu) = \psi(\mu)\,.\,m'$. This completes the proof of (2.8). □

The relationship between our affine Temperley-Lieb algebras and the quotient of the Hecke algebra discussed in §0 is explained in the next result.

(2.9) PROPOSITION (cf. [FG]). *There is an algebra homomorphism $\rho\colon H_n^a(q) \to \mathbf{T}^a(n)$ (see (0.1)) which takes T_i to $-f_i - q^{-1}$ for $i = 1, \ldots, n$. The kernel of ρ is the ideal I_n^a of (0.5), while the image of ρ is spanned by non-monic diagrams : $n \to n$ of even rank, together with the identity. Thus the latter diagrams span an algebra which is isomorphic to $TL_n^a(\delta)$ (see (0.5)).*

Idea of proof. Write $C_i = -(T_i + q^{-1})$. Then $H_n^a(q)$ is generated as R-algebra by $C_1, \ldots, C_n$ subject to the relations

$$\begin{gathered}C_iC_j = C_jC_i \quad \text{if } |i-j| \geq 2 \text{ and } (i,j) \neq (1,n)\\ C_i^2 = \delta C_i\\ C_iC_{i+1}C_i - C_i = C_{i+1}C_iC_{i+1} - C_{i+1} = -q^{-3}E_i\,, \text{ where } E_i \text{ is as in §0}\end{gathered}$$

and the index i is taken modulo n in the last equation. One checks easily that the diagrams f_i satisfy these relations with E_i replaced by 0, whence the indicated map defines a homomorphism of algebras. The kernel contains the E_i, and hence contains I_n^a. The image is the algebra generated by the f_i, which is identified as in the statement of (2.9) by Corollary (1.9). To see that the kernel is no larger than I_n^a, we refer the reader to [FG]. □

The modules $W_{t,z}(n)$ and $L_{t,z}(n)$ may be regarded as modules for the subalgebra TL_n^a of $\mathbf{T}^a(n)$. It is a simple consequence of (2.9) that as TL_n^a-modules $W_{t,z} \cong W_{t,y}$ if $t = n$ or $z + y = 0$. Moreover the argument of [GL 2.6] shows that $L_{t,z}$ remains irreducible as a TL_n^a-module, unless $t = 0$ and $z^2 = -1$. In the remaining case, $W_{0,z}$ is the direct sum of two submodules $W_{0,z}^+$ and $W_{0,z}^-$ spanned respectively by the even and odd standard diagrams $: 0 \to n$. If $q^2 \neq -1$ these modules have irreducible heads $L_{0,z}^+$ and $L_{0,z}^-$ whose sum is $L_{0,z}$. This leads to the following description of the cell modules and irreducible modules for the algebra TL_n^a.

(2.9.1) COROLLARY. *Let* $\overline{\Lambda}^a(n)$ *be the quotient of* $\Lambda^a(n)$ *by the equivalence relation* $(t,z) \equiv (t,y)$ *if* $t = n$ *or* $z = -y$, *with new points* $(0,z)^+$ *and* $(0,z)^-$ *replacing* $(0,z) \in \Lambda^a(n)$ *if* n *is even and* $z^2 = -1$. *Then Theorem (2.8) applies to* $W_{t,z}(n)$ *and* $L_{t,z}(n)$ *regarded as* TL_n^a*-modules, with* $\overline{\Lambda}^a(n)$ *replacing* $\Lambda^a(n)$ *and the representations being realised as above.*

(2.10) THE JONES ANNULAR ALGEBRAS

The *Brauer centraliser algebra* is the free R-module $\mathbf{B}(n)$ generated by fixed point free (but not necessarily planar) involutions ϕ of $\mathbf{n\#n}$ with multiplication defined analogously to (1.2) and (2.1). There is a unique algebra homomorphism: $\psi\colon \mathbf{T}^a(n) \to \mathbf{B}(n)$ which takes an (affine) diagram $\alpha\colon n \to n$ to $(\delta)^{g(\alpha)}$ times the involution ψ_α of $\mathbf{n\#n}$ which interchanges the vertex $\ell(x)$ or $u(x)$ with $\ell(y)$ or $u(y)$ $(x, y \in \mathbf{n})$ precisely when ϕ_α maps $\ell(i,x)$ or $u(i,x)$ to $\ell(j,y)$ or $u(j,y)$ for some $i,j \in \mathbf{Z}$. *Jones' annular algebra* $\mathbf{J}(n)$ is the image of this algebra homomorphism. This algebra is known to have a cellular structure [GL]; the associated cell modules are related to those of $\mathbf{T}^a$ as follows. If $(t,z) \in \Lambda^a(n)$ is such that $t > 0$ and $z^t = 1$, then the kernel of ψ annihilates the $\mathbf{T}^a(n)$-module $W_{t,z}(n)$, and so we obtain $\mathbf{J}(n)$-modules $W_{t,z}(n)$ and $L_{t,z}(n)$; with the notation of [GL] the first module is (canonically isomorphic to) the cell representation $W(t,z)$ while the second is its unique irreducible head $L(t,z)$. The remaining cell representation $W(0,1)$ of $\mathbf{J}(n)$ as defined in [GL], is the quotient $W_{0,q}(n)/M$ where M is the image of the map $\theta_n\colon W_{2,1}(n) \to W_{0,q}(n)$ of Theorem (3.4) below. The unique head $L(0,1)$ of $W(0,1)$ is $L_{0,q}(n)$.

It is therefore clear that the representation theory of the Jones algebra $J(n)$ is included in the representation theory of our affine algebras. Its cell representations form a subset of those of $\mathbf{T}^a(n)$, with one exception.

The (finite) Temperley-Lieb category $\mathbf{T}$ is a subcategory of the affine Temperley-Lieb category $\mathbf{T}^a$. Therefore the cell representations of $\mathbf{T}^a$ give rise to representations of $\mathbf{T}$ by restriction. We complete this section by describing the structure of the resulting restricted $\mathbf{T}(n)$ modules, as well as some "asymptotic" ones.

(2.11) LEMMA. *Let R be a ring with an invertible element q. Consider the affine Temperley-Lieb category $\mathbf{T}^a = \mathbf{T}^a_{R[z,z^{-1}],q}$ over the ring $R[z,z^{-1}]$ of Laurent polynomials in an indeterminate z and let $t \in \mathbf{Z}_{\geq 0}$. Define coefficient functions $r^\mu_\nu(x) \in R[z,z^{-1}]$ of the cell representation $W_{t,z}$ by:*

$$x * \mu = \sum_{\substack{\nu:\, t \to s \\ \text{standard}}} r^\mu_\nu(x)\nu \quad \text{in } W_{t,z}(s)$$

where $\mu\colon t \to n$ and $\nu\colon t \to s$ are standard (affine) diagrams and $x \in \mathbf{T}^a(n,s)$. If x is a finite diagram α, then the coefficient $r^\mu_\nu(\alpha)$ vanishes unless $l = |\mu| - |\nu| \geq 0$ and $\nu \circ \tau^i_t = \alpha \circ \mu$ for some $i \in \mathbf{Z}$. In this case $r^\mu_\nu(\alpha)z^l$ is a polynomial in $R[z^2]$ of degree at most l. Furthermore if μ_1 and μ_2 are standard diagrams from t to n, $\langle \mu_1, \mu_2 \rangle_{t,z} z^{|\mu_1|+|\mu_2|}$ also lies in $R[z^2]$.

Proof. Although these statements are staightforward consequences of Lemma (1.5), we provide the details for the reader's convenience. Recall that $\alpha * \mu$ is equal to $\alpha \circ \mu$ if this has t through strings, and is zero otherwise. In the former case, $\alpha \circ \mu = \nu \circ \tau^j_t$ for some standard ν and $j \in \mathbf{Z}$, and we have

$$\alpha * \mu = \delta^{m(\alpha,\mu)} y^j \nu$$

where $y = z$ if $t > 0$ and $y = z + z^{-1}$ if $t = 0$. Thus the coefficient $r^\mu_\nu(\alpha)$ vanishes except for this particular standard diagram ν, and $r^\mu_\nu(\alpha) = \delta^{m(\alpha,\mu)} y^j$.

We now relate the ranks to j. Since $|\alpha| = 0$ by (1.5) we have $|\alpha \circ \mu| \leq |\mu|$ and these have the same parity. Since ν is standard, we also have $|\nu \circ \tau^j| = |j| + |\nu|$ whence $|j| \leq |\mu| - |\nu| = l$ and still both sides have the same parity. Since the left hand side is nonnegative, so is the right hand side. Furthermore, $r^\mu_\nu(\alpha)$ is an R-linear combination of integer powers of z, all of which have the parity of j, the smallest of which is $-j$ and the largest being j. It follows that $r^\mu_\nu(\alpha)z^l$ is a polynomial in z^2, of degree $(j+l)/2 \leq l$, which proves the first statement.

Next we compute the bilinear pairing. Let μ_1 and $\mu_2 : t \to n$ be standard. Their scalar product vanishes, unless $\mu_2^* \circ \mu_1 : t \to t$ has t through strings. In this case we have $\mu_2^* \circ \mu_1 = \tau_t^k$ for some $k \in \mathbf{Z}$ and

$$\langle \mu_1, \mu_2 \rangle_{t,z} = \delta^{m(\mu_2^*,\mu_1)} y^k$$

where y is as above. The proof is now completed as above, bearing in mind that by (1.5) $|k| \le |\mu_1| + |\mu_2|$ and the two sides have the same parity. □

As a consequence of Lemma (2.11), we may construct $\mathbf{T}$-modules $W_{t,0}$ and $W_{t,\infty}$ as follows. If $n \in \mathbf{Z}_{\ge 0}$, let $W_{t,0}(n)$ be the free R-module generated by standard affine diagrams (see (1.7)) $\mu : t \to n$. If $\alpha : n \to s$ is a finite diagram and $\mu : t \to n$ is standard, define

(2.11.1) $$\alpha * \mu = r_\nu^\mu(\alpha)_0 \nu$$

where ν is as in (2.11) and $r_\nu^\mu(\alpha)_0$ is the constant term of $r_\nu^\mu(\alpha) z^l$. Extend this R-bilinearly to an action of $\mathbf{T}(n)$. The module $W_{t,\infty}$ is constructed in analogous fashion using the constant term of $r_\nu^\mu(\alpha) z^{-l}$ in $R[z^{-2}]$. One then has an invariant pairing

(2.11.2) $$\langle\ ,\ \rangle_{t,0} : W_{t,0}(n) \times W_{t,\infty}(n) \to R$$

where $\langle \mu, \nu \rangle_{t,0}$ is the constant term of $\langle \mu, \nu \rangle_{t,z} z^{|\nu|+|\mu|}$.

(2.12) THE FINITE TEMPERLEY-LIEB ALGEBRAS

We shall describe how the affine modules are related to the finite modules of [GL,§6] (see (2.2) above). Let $t \in \mathbf{Z}_{\ge 0}$ and let z be 0, ∞, or an invertible element of R. We construct a filtration of the $\mathbf{T}$-module $W_{t,z}$ whose quotients are cell representations. If $s \in \mathbf{Z}_{\ge 0}$ is such that $s \equiv t \mod 2$, then for each $n \in \mathbf{Z}_{\ge 0}$ let $W_{t,z}^s(n)$ be the R-span of the standard affine diagrams $\mu : t \to n$ of rank $|\mu| < (s-t)/2$. This defines an increasing family

$$0 = W_{t,z}^t(n) \subset W_{t,z}^{t+2}(n) = W_t(n) \subset W_{t,z}^{t+4}(n) \subset \cdots \subset W_{t,z}^n(n) \subset W_{t,z}^{n+2}(n) = W_{t,z}(n)$$

of $\mathbf{T}$-submodules of $W_{t,z}(n)$ (of course $W_{t,z}^s(n) = 0$ for $s \le t$ and $W_{t,z}^s(n) = W_{t,z}(n)$ for $s > n$). It follows from (1.9.1) that there is an exact sequence of natural transformations:

(2.12.1) $$0 \longrightarrow W_{t,z}^s \longrightarrow W_{t,z}^{s+2} \longrightarrow W_s \longrightarrow 0$$

where the left map is inclusion and the right map is given (cf. (1.9.1)) at n by:

$$W_{t,z}^{s+2}(n) \to W_s(n) : \mu \circ \eta^{(s-t)/2} \mapsto \mu .$$

(2.13) THE TRIVIAL REPRESENTATION OF THE FINITE TEMPERLEY-LIEB ALGEBRAS

The cell module $W_s(s)$ is one-dimensional and will be referred to as the *trivial representation* of $\mathbf{T}(s)$. Observe that the diagrams $f_i \in \mathbf{T}(s)$ all act as the zero operator in this representation, whence if e_s is the corresponding idempotent in $\mathbf{T}(s)$ (e_s exists generically, by generic semisimplicity), then $f_i * e_s = 0 = e_s * f_i$ for all i. The idempotent e_s is referred to in the literature (cf. [MV], [We], [Li] and [J3], where e_s was first identified) as the Jones, or augmentation idempotent of $\mathbf{T}(s)$.

(2.14) LEMMA. *Let t, s and k be non-negative integers such that $s = t + 2k$. If $x \in W_{t,z}(s)$ is annihilated by all finite diagrams $\alpha: s \to s$ except id_s, then x is a scalar multiple of $e_s * \eta^k$, where e_s is defined above and η^k is defined in (1.9.1).*

Proof. We may suppose that $k > 0$, since the case $k = 0$ is trivial. The hypothesis implies that Rx is a realization of the trivial representation of $\mathbf{T}(s)$, whence $x \in e_s * W_{t,z}(s)$. We shall therefore be done if we show that

(2.14.1) $$e_s * W_{t,z}(s) = Re_s * \eta^k .$$

Now η^k is characterised among the standard diagrams $: t \to s$ as the unique diagram of maximal rank (k). If $\mu: t \to s$ is standard and $|\mu| < k$, then $\mu = f_i * \nu$, for some standard diagram $\nu: t \to s$ and $i \in \{1, 2, \dots, s-1\}$ because ϕ_μ must interchange two upper vertices in the fundamental rectangle (recall $k > 0$). Hence $e_s * \mu = e_s * f_i * \nu = 0$, proving (2.14.1) and hence the lemma. □

§3. HOMOMORPHISMS AND NATURAL TRANSFORMATIONS

For any integer n, define the Gaussian integer $[n]_{\mathrm{x}}$ in the function field $\mathbf{Q}(\mathrm{x})$ by

$$[n]_{\mathrm{x}} := \frac{\mathrm{x}^n - \mathrm{x}^{-n}}{\mathrm{x} - \mathrm{x}^{-1}} = \mathrm{x}^{n-1} + \mathrm{x}^{n-3} + \cdots + \mathrm{x}^{1-n} .$$

Define the Gaussian x-factorial by

$$[n!]_{\mathrm{x}} = [n]_{\mathrm{x}}[n-1]_{\mathrm{x}} \dots [2]_{\mathrm{x}}[1]_{\mathrm{x}} .$$

For any pair $n \geq k$ of positive integers, the Gaussian binomial coefficient is