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Artikel: THE REPRESENTATION THEORY OF AFFINE TEMPERLEY-LIEB ALGEBRAS

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Kapitel: §5. DECOMPOSITION MATRICES

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(4.8) COROLLARY.

(1) *If n is an odd positive integer, then Jones' annular algebra* $\mathbf{J}(n)$ *(with parameter $\delta = -q - q^{-1}$) is non-semisimple if and only if there exist distinct odd integers $s, t \in \mathbf{n}$ such that $q^{st} = 1$.*

(2) *If n is an even positive integer, then Jones' annular algebra* $\mathbf{J}(n)$ *(with parameter $\delta = -q - q^{-1}$) is non-semisimple if and only if $q^{\frac{n}{2}+1} = 1$ or there exist distinct even integers $s, t \in \mathbf{n}$ such that $q^{\frac{st}{2}} = 1$.*

Proof. By [GL, 3.8] the algebra is semisimple precisely when the bilinear pairing $\langle\ ,\ \rangle_{t,z}$ is non-degenerate on each cell representation (of $\mathbf{J}(n)$); this condition is equivalent to the vanishing of the determinant $\det G_{t,z}(n)$, which by (4.7) immediately yields the stated condition. $\square$

§5. DECOMPOSITION MATRICES

(5.1) THEOREM. *Let R be an algebraically closed field of characteristic zero and q a nonzero element of R. Let $\preceq$ be the weakest partial order on the set Λ^a defined in (2.6) such that $(t, z) \preceq (s, y)$ if (t, z) and (s, y) satisfy the hypotheses of Theorem (3.4) for q or q^{-1}. If $(t, z) \in \Lambda^a$, $n \in \mathbf{Z}_{\geq 0}$ and $(s, y) \in \Lambda^a(n)$, then the multiplicity of the irreducible $\mathbf{T}^a(n)$-module $L_{s,y}(n)$ in the cell representation $W_{t,z}(n)$ of (2.6) is one if $(s, y) \succeq (t, z)$ and zero otherwise.*

Proof. Let R be a field and $q \in R$. Let $p\colon R[\mathrm{y}] \to R$ be the R-algebra homomorphism defined by $\mathrm{y} \mapsto q + q^{-1}$, where y is an indeterminate over R. Suppose W is a free $R[\mathrm{y}]$-module of finite rank with an $R[\mathrm{y}]$-bilinear form $\langle\ ,\ \rangle\colon W \times W \to R[\mathrm{y}]$. If R is regarded as a $R[\mathrm{y}]$-module via the homomorphism p, the free R-module $W_R = R \otimes_{R[\mathrm{y}]} W$ inherits an R-bilinear form $\langle\ ,\ \rangle_R\colon W_R \times W_R \to R$ given by $\langle 1 \otimes x, 1 \otimes y\rangle_R = p(\langle x, y\rangle)$. Choose $R[\mathrm{y}]$-bases B_1 and B_2 of W and let G denote the associated gram matrix of $\langle\ ,\ \rangle$. If this form is nonsingular (i.e. $\det G \neq 0$), then it may be shown that the multiplicity of the polynomial $\mathrm{y} - q - q^{-1}$ in the determinant $\det G$ is greater than or equal to the R-dimension of the radical of $\langle\ ,\ \rangle_R$. In fact if we denote the multiplicity of the polynomial $\mathrm{y} - q - q^{-1}$ in $f \in R[\mathrm{y}]$ by $\mathrm{mult}(f)$, then

$$\mathrm{mult}(\det G) = \sum_{i>0} \dim \mathrm{rad}^i$$

where rad^i denotes the image under $\phi\colon W \to W_R : w \mapsto 1 \otimes w$ of the $R[\mathrm{y}]$-submodule $\{w \in W \mid \langle w, v\rangle \in (\mathrm{y} - q - q^{-1})^i R[\mathrm{y}]$ for any $v \in W\}$.

(Since $R[\mathrm{y}]$ is a principal ideal domain, row and column operations may be used to reduce the proof of this fact to the easy case when G is diagonal.) We shall use this elementary result to give a bound for the dimension of the radical of the restriction of $\langle\ ,\ \rangle_{t,z}$ to $W^s_{t,z}(n)$.

Let $t \leq s$ be non-negative integers of the same parity, $n \in \mathbf{Z}_{\geq 0}$ and assume the hypotheses of the statement. Consider $\mathbf{T}^a_{(R[\mathrm{x}],-\mathrm{x})}$. We shall compute the determinant of the gram matrix $G^s_{t,0}(n)$ as a polynomial in $\mathrm{y} = \mathrm{x} + \mathrm{x}^{-1}$. Our first goal is to compute the multiplicity of $\mathrm{y} - q - q^{-1}$ in this polynomial, i.e. to compute $\operatorname{mult}(\det G^s_{t,0}(n))$. Let l denote the order of q^2. Since $[n]_{\mathrm{x}}$ and $\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]_{\mathrm{x}}$ are polynomials in $\mathrm{y} = \mathrm{x} + \mathrm{x}^{-1}$ we may speak of the multiplicity of $\mathrm{y} - q - q^{-1}$ in these polynomials and it is straightforward that

$$\operatorname{mult}\,[n]_{\mathrm{x}} = \begin{cases} 1 & \text{if } l \neq 1, \infty \text{ and } l \text{ divides } n, \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{and hence} \quad \operatorname{mult}\begin{bmatrix} n \\ i \end{bmatrix}_{\mathrm{x}} = \begin{cases} 1 & \text{if } l \neq \infty \text{ and } \operatorname{res}_l(n) < \operatorname{res}_l(i), \\ 0 & \text{otherwise,} \end{cases}$$

where $\operatorname{res}_l(n) \in \{0, 1, \ldots, l-1\}$ is determined by $\operatorname{res}_l(n) \equiv n \mod l$.

We next give an expression for $\operatorname{mult}([t; r]_{\mathrm{x}}/[s; r]_{\mathrm{x}})$. Let $r \geq s$ have the same parity as s (or t) and write $X = \{0, 1, \ldots, l-1\}$. Then there exist unique elements $k \in \mathbf{Z}$ and $\bar{r} \in X$ such that $r = kl + \bar{r}$. Let $\bar{t}$ denote the unique element of X such that $kl + \bar{t} \equiv \pm t \mod 2l$; define $\bar{s}$ similarly. Define:

$$\epsilon^s_t(r) = \begin{cases} 1 & \text{if } \bar{s} \leq \bar{r} < \bar{t}, \\ -1 & \text{if } \bar{t} \leq \bar{r} < \bar{s}, \\ 0 & \text{otherwise.} \end{cases}$$

The function $\epsilon^t_s(r)$ satisfies

(1) $\epsilon^t_s(r) = \epsilon^{-t}_s(r) = \epsilon^{t+2l}_s(r)$

(2) $\epsilon^s_t(r) = -\epsilon^t_s(r)$.

It is easy to see that if $0 \leq t \leq s \leq r$, then

$$\epsilon^s_t(r) = \operatorname{mult}([t; r]_{\mathrm{x}}/[s; r]_{\mathrm{x}}).$$

By Corollary (4.5) and Proposition (4.6), we have

$$\operatorname{mult}(\det G^s_{t,0}(n)) = \sum_{\substack{r \geq s \\ r \equiv t \mod 2}} \epsilon^s_t(r) \dim W_r(n). \tag{5.1.1}$$

If $l = \infty$ or $s \equiv t$ or $-t \mod 2l$, then $\epsilon^s_t(r) = 0$ and so the multiplicity (5.1.1) is zero. For the remainder of this paragraph, assume that $l \neq \infty$ and

$s \not\equiv \pm t \mod 2l$. Let $t' \in \mathbf{Z}$ be minimal such that $t' > s$ and $t' \pm t \equiv 0 \mod 2l$. Let $s' \in \mathbf{Z}$ be maximal such that $t' > s'$ and $s' \pm s \equiv 0 \mod 2l$. Then $s + 2l > t' > s' \geq s > t$. Now in order to compute $\operatorname{mult}(\det G^s_{t,0}(n))$, we partition the sum on the right side of (5.1.1) into three parts:

(1) $s \leq r < s'$.

(2) $s' \leq r < t'$.

(3) $t' \leq r$.

For the terms in the first part, $\epsilon^s_t(r) = 0$. For those in the second part $\epsilon^s_t(r) = 1$ and consequently, these terms contribute $\dim W^{t'}_{s',0}(n) = \sum_{s' \leq r < t'} \dim W_r(n)$ to the sum. The terms in the third part have $\epsilon^s_t(r) = -\epsilon^{t'}_{s'}(r)$ (by properties (1) and (2) of the function $\epsilon^s_t(r)$) and so these terms contribute $\operatorname{mult}(\det G^{t'}_{s',0}(n))$ to the sum.

It follows that

$$\text{(5.1.2)} \qquad \operatorname{mult}(\det G^s_{t,0}(n)) = \dim W^{t'}_{s',0}(n) - \operatorname{mult}(\det G^{t'}_{s',0}(n)).$$

Note that equation (5.1.2) should be interpreted as a recurrence relation for $\operatorname{mult}(\det G^s_{t,0}(n))$, which together with the initial condition $\operatorname{mult}(\det G^s_{t,0}(n)) = 0$ if $n \leq t$, determines the multiplicity.

Now fix $n \in \mathbf{Z}_{\geq 0}$. Choose $(t, z) \in \Lambda^a$ such that $t \leq n$ and $t \equiv n \mod 2$. To prove the Theorem, we shall construct a composition series for $W_{t,z}(n)$.

If (t, z) is maximal in $\Lambda^a(n)$ (with respect to $\prec$), then it follows from Corollary 4.4 and Proposition 4.6, that $\operatorname{rad}_{t,z}(n) = 0$; hence the irreducible module $L_{t,z}(n)$ coincides with $W_{t,z}(n)$ and the statement follows.

Assume that (t, z) is not a maximal element of $\Lambda^a(n)$ and choose $(s, y) \in \Lambda^a(n)$ such that $(s, y) \succ (t, z)$ and s is minimal with respect to this property. Then the hypotheses of Theorem (3.4) are satisfied (possibly after replacing q by q^{-1}) and so we have an injective homomorphism $\theta_n \colon W_{s,y}(n) \to W_{t,z}(n)$ of $\mathbf{T}^a_{R,q}(n)$-modules. The quotient $Q = W_{t,z}(n)/\operatorname{Im}\theta_n$ has basis $\mu + \operatorname{Im}\theta_n$ indexed by standard diagrams $\mu \colon t \to n$ of rank strictly less than $(s-t)/2$. By (2.8), the image of θ_n is contained in $\operatorname{rad}_{t,z}(n)$, whence the bilinear form $\langle\ ,\ \rangle_{t,z}$ descends to $Q \times Q \to R$; its gram matrix (with respect to the basis above) is $G^s_{t,z}(n)$ and $L_{t,z}(n)$ is the quotient of Q by its radical which we denote by $\operatorname{rad}^s_{t,z}(n)$. Consider, for the moment, $\mathbf{T}^a_{R[x],x}$. The multiplicity $\operatorname{mult}(\det G^s_{t,z}(n)) = \operatorname{mult}(\det G^s_{t,0}(n))$ by Corollary (4.4); it follows from the remarks concerning linear algebra at the beginning of this proof that

$$\text{(5.1.3)} \qquad \dim \operatorname{rad}^s_{t,z}(n) \leq \operatorname{mult}(\det G^s_{t,0}(n)).$$

If the order l (of q^2) is infinite, then (s,y) is the unique element of Λ^a such that $(s,y) \succ (t,z)$. If l is finite and $s \equiv t$ or $-t \mod 2l$, then (s,y) is the unique element of Λ^a which covers (t,z). In either case, we saw above that $\mathrm{mult}(\det G^s_{t,0}(n)) = 0$ and so $\mathrm{rad}^s_{t,z}(n) = 0$. Therefore $Q = L_{t,z}(n)$ and the composition factors of $W_{t,z}(n)$ are $L_{t,z}(n)$ together with those of $W_{s,y}(n)$, as required.

Assume that l is finite and $s \not\equiv \pm t \mod 2l$. Let s' and t' be as above and $y' = \epsilon y^{-1}$ where $\epsilon = q^{(s+s')/2} = \pm 1$. Then (s', y') is the unique element of Λ^a such that $(s', y') \succ (t,z)$ and $(s', y') \not\succeq (s,y)$. If $s' > n$, then the initial condition associated with (5.1.2) shows that $\mathrm{mult}(\det G^s_{t,0}(n)) = 0$ and so $\mathrm{rad}^s_{t,z}(n) = 0$; hence $Q = L_{t,z}(n)$ and the statement of (5.1) follows as in the previous paragraph.

Finally, assume that $s' \le n$. By Theorem (3.4) (with q^{-1} replacing q), there exists an injective $\mathbf{T}^a(n)$-homomorphism $\theta'_n \colon W_{s',y'}(n) \to W_{t,z}(n)$. Thus $L_{s',y'}(n)$ is a composition factor of $W_{t,z}(n)$. Arguing by induction in the poset Λ^a, we may assume that $L_{s',y'}(n)$ is not a composition factor of $W_{s,y}(n) \cong \mathrm{Im}(\theta_n)$ since $(s', y') \not\succeq (s,y)$. It follows that the irreducible module $L_{s',y'}(n)$ is a composition factor of $\mathrm{rad}^s_{t,z}(n)$ and we have, using (5.1.3),

$$\dim L_{s',y'}(n) \le \dim \mathrm{rad}^s_{t,z}(n) \le \mathrm{mult}(\det G^s_{t,0}(n)).$$

Arguing as above with (s', y') in place of (t,z) we have

$$\dim L_{s',y'}(n) = \dim Q' - \dim(\mathrm{rad}^{t'}_{s',y'}(n)) \ge \dim W^{t'}_{s',y'}(n) - \mathrm{mult}(\det G^{s'}_{t',0}(n)).$$

Now (5.1.2) asserts that the two ends of this chain of inequalities are equal. Hence we have equality at every step and in particular $L_{s',y'}(n)$ is isomorphic to $\mathrm{rad}^s_{t,z}(n)$. Thus the composition factors of $W_{t,z}(n)$ are $L_{t,z}(n)$ (if $q^2 \neq 0$ or $(t,z) \neq (0,q)$) and $L_{s',y'}(n)$ together with those of $W_{s,y}(n)$, as required. □

(5.2) COROLLARY. *Assume the hypotheses and notation of Theorem 5.1 and let* $\mathbf{J}(n)$ *be Jones' annular algebra (see (2.10)). If* $(t,z) \in \Lambda^a(n)$ *is such that* $t > 0$ *and* $z^t = 1$, *then the* $\mathbf{J}(n)$*-module* $W_{t,z}(n)$ *has composition factors* $L_{s,y}(n)$ *indexed by* $(s,y) \in \Lambda^a(n)$ *such that* $(s,y) \succeq (t,z)$. *The remaining cell module* $W_{0,q}/M$ *(2.10) has composition factors* $L_{s,y}(n)$ *indexed by* $(s,y) \in \Lambda^a(n)$ *such that* $(s,y) \succeq (0,q)$ *and* $(s,y) \not\succeq (2,1)$.

The next result is implicit in [DJ] and may be found in [Ma], [GW] and [W].

(5.3) THEOREM. *Let R be a field of characteristic zero, let q be a nonzero element of R and let $\mathbf{T}(n) = \mathbf{T}_{R,q}(n)$ be the Temperley-Lieb algebra over R, with parameter q. If $n, t \in \mathbf{Z}_{\geq 0}$ and $s \in \Lambda(n)$ (2.3) then the multiplicity of the irreducible $\mathbf{T}(n)$-module $L_s(n)$ in the cell representation $W_t(n)$ (2.2) is one if*

(1) $s = t$, *or*

(2) q^2 *has finite order* l, $t + 2l > s > t$ *and* $s + t + 2 \equiv 0 \mod 2l$,

and zero otherwise.

Proof. Adopt the notation of the proof of (5.1). Let $t \in \Lambda(n)$ and note that $G_t(n) = G_t^{t+2}(n)$. If there is no element $s \in \Lambda(n)$ such that (2) holds, then the computations above show that $\operatorname{mult}(\det G_t(n)) = 0$; hence $W_t(n)$ is irreducible and the statement follows. If q^2 has finite order l and $s \in \Lambda(n)$ satisfies (2), then Corollary (3.5) provides a nonzero homomorphism of $\mathbf{T}(n)$-modules $\theta_n \colon W_s(n) \to W_t(n)$. Hence $L_s(n)$ is a composition factor of $W_t(n)$ and we have

$$\dim L_s(n) \leq \dim \operatorname{rad}_t(n) \leq \operatorname{mult}(\det G_t(n))$$

as in the previous proof. However,

$$\dim L_s(n) = \dim W_s(n) - \dim \operatorname{rad}_s(n) \geq \dim W_s(n) - \operatorname{mult}(\det G_s(n)).$$

Now (5.1.2) again asserts that the ends of this chain of inequalities are equal. Therefore we have equality at each step and in particular $L_s(n)$ is isomorphic to $\operatorname{rad}_t(n)$. $\square$

(5.4) REMARKS.

(1) The decomposition matrices in Theorems (5.1) and (5.3) are "independent of n"; one may therefore speak of the multiplicity of $L_{s,y}$ in $W_{t,z}$ and of L_s in W_t.

(2) Since the dimension of $W_{t,z}(n)$ is known (1.12), the multiplicities of (5.1) may be used to give formulae for the dimensions of the irreducible modules $L_{t,z}(n)$. These formulae are just the inversions of the relations

$$\binom{n}{(n-t)/2} = l_{t,z}(n) + \sum_{\substack{(s,y)\in\Lambda^a \\ (s,y)\succ(t,z)}} l_{s,y}(n)$$

where $l_{s,y}(n) = \dim L_{s,y}(n)$. A similar remark applies to the dimensions of the irreducible modules for the Jones and Temperley-Lieb algebras.

(3) The proofs of (5.1) and (5.3) yield the radical series of the modules concerned; $L_{s,y}(n)$ lies in the k-th layer of $W_{t,z}(n)$ if the length of the interval between (s, y) and (t, z) in Λ^a is k. One might expect the layers of the radical series of the cell modules to coincide with the layers (denoted rad^i above) of some "Jantzen filtration" of the cell representation and its bilinear form (after scaling the indices).

(4) If the characteristic of R times the order l of q^2 exceeds the cardinality of n then Theorems (5.1) and (5.3) remain valid without the restriction that R have characteristic zero.

(5) As indicated in (2.9.1), all of our results may be interpreted as statements about the representation theory of TL_n^a; in particular, they illuminate a part of the modular representation theory of the affine Hecke algebra $H_n^a(q)$. One could ask which irreducible representations of the affine Hecke algebra correspond in the Kazhdan-Lusztig parametrization [KL2] to our $L_{t,z}$. A similar comment applies to the connection with the work [Gj].

REFERENCES

[Ch] CHEREDNIK, I. V. A new interpretation of Gel′fand Tzetlin bases. *Duke Math. J. 54* (1987), 563–577.

[DJ] DIPPER, R. and G. JAMES. The q-Schur algebra. *Proc. London Math. Soc. (3) 59* (1989), no. 1, 23–50.

[FG] FAN, C. K. and R. M. GREEN. On the affine Temperley-Lieb algebras. Preprint.

[FY] FREYD, P. J. and D. N. YETTER. Braided compact closed categories with applications to low-dimensional topology. *Adv. Math. 77* (1989), 156–182.

[Gj] GROJNOWSKI, I. Representations of affine Hecke algebras (and affine quantum GL_n) at roots of unity. *Internat. Math. Res. Notices*, 1994, no. 5, 215ff.

[GL] GRAHAM, J. J. and G. I. LEHRER. Cellular Algebras. *Invent. Math. 123* (1996), 1–34.

[Gr] GRAHAM, J. J. PhD Thesis. Sydney University, 1995.

[GW] GOODMAN, F. M. and H. WENZL. The Temperley-Lieb algebra at roots of unity. *Pacific J. Math. 161* (1993), 307–334.

[J1] JONES, V. F. R. Hecke algebra representations of braid groups and link polynomials. *Ann. Math. 126*, (1987), 335–388.

[J2] —— A quotient of the affine Hecke algebra in the Brauer algebra. *L'Enseignement Math. (2) 40* (1994), 313–344.

[J3] —— Index for subfactors. *Invent. Math. 72* (1983), 1–25.

[Ja] JAMES, G. *Representations of General Linear Groups*. London Math. Soc. Lect. Note Series 94. Cambridge University Press, Cambridge, 1984.

[KL1] KAZHDAN, D. and G. LUSZTIG. Equivariant K-theory and representations of Hecke algebras. *Proc. Amer. Math. Soc. 94* (1985), 337–342.