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Artikel: NEW PROOF OF VINCENT'S THEOREM

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6. VINCENT'S PROOF REVISITED IN MODERN TERMS

The 'qualitative' argument used by Vincent to prove his theorem can easily be recast in modern terms to obtain Uspensky's result, but under the only condition that

$$F_h F_{h-1} \Delta > 1 + \frac{1}{\varepsilon_n} .$$

In view of the fact that Viète's formulae relate the coefficients of a polynomial to its roots, it is far from astonishing that Vincent's proof can be improved to provide a quantitative estimate for h. But it is worthwhile to observe that it gives exactly Uspensky's result.

Consider once again the proof of Theorem 4.1 up to (4.3), which describes the polynomial $G(x)$.

Factoring out $g(b)$ we have

$$G(x) = (x+1)^{n-1} g(b) \left[1 + \frac{g'(b)}{1! \cdot g(b)} u + \frac{g''(b)}{2! \cdot g(b)} u^2 + \cdots + \frac{g^{(n-1)}(b)}{(n-1)! \cdot g(b)} u^{n-1} \right].$$

Recalling that $u = \dfrac{a-b}{1+x}$, we have

$$\text{(6.1)} \qquad \frac{G(x)}{g(b)} = (x+1)^{n-1} + \frac{g'(b)}{1! \cdot g(b)} (a-b)(x+1)^{n-2} + \cdots + \frac{g^{(n-1)}(b)}{(n-1)! \cdot g(b)} (a-b)^{n-1} .$$

Since the roots of $g(x)$ are $x_1, x_2, \ldots, x_{n-1}$, we have

$$\frac{g'(x)}{1! \cdot g(x)} = \sum_i \frac{1}{1!} \frac{1!}{x - x_i} = \sum_i \frac{1}{x - x_i} ,$$

$$\frac{g''(x)}{2! \cdot g(x)} = \sum_{i,j} \frac{1}{2!} \frac{2!}{(x - x_i)(x - x_j)} = \sum_{i,j} \frac{1}{(x - x_i)(x - x_j)} ,$$

$$\frac{g'''(x)}{3! \cdot g(x)} = \sum_{i,j,k} \frac{1}{3!} \frac{3!}{(x - x_i)(x - x_j)(x - x_k)} = \sum_{i,j,k} \frac{1}{(x - x_i)(x - x_j)(x - x_k)} ,$$

. . . .

The above sums contain respectively $\binom{n-1}{1}$, $\binom{n-1}{2}$, $\binom{n-1}{3}$, ... terms.

Since $F_h F_{h-1} \Delta > 1 + \dfrac{1}{\varepsilon_n} > 1$, we have, in particular, $|b - a| < \Delta$. Hence

$$|b-a| = \theta \cdot \Delta, \quad \text{with} \quad \theta < 1 .$$

Observe that

$$|b-x_i| = |b-x_0+x_0-x_i| > |x_0-x_i| - |b-x_0| > \Delta - \theta\Delta = (1-\theta)\Delta ;$$

hence

$$\left| \frac{g^{(k)}(b)}{k! \cdot g(b)} \right| < \binom{n-1}{k} \frac{1}{(1-\theta)^k \Delta^k} ,$$

and

$$\left| \frac{g^{(k)}(b)}{k! \cdot g(b)} (a-b)^k \right| < \binom{n-1}{k} \frac{1}{\Delta^k} \Delta^k \cdot \frac{\theta^k}{(1-\theta)^k} = \binom{n-1}{k} (\frac{\theta}{1-\theta})^k .$$

Let $\dfrac{\theta}{1-\theta} = \tau$. The absolute value of the coefficient of x^i on the right hand side of (6.1) is smaller than

$$\binom{n-1}{i} + \binom{n-1}{1}\binom{n-2}{i}\tau + \binom{n-1}{2}\binom{n-3}{i}\tau^2$$
$$+ \binom{n-1}{3}\binom{n-4}{i}\tau^3 + \cdots = \sum_{k=0}^{n-1-i} \binom{n-1}{k}\binom{n-1-k}{i}\tau^k$$
$$= \binom{n-1}{i}(1+\tau)^{n-1-i} = \binom{n-1}{i}(\frac{1}{1-\theta})^{n-1-i} .$$

To apply Lemma 5.1 we need to impose the condition

$$\left| (\frac{1}{1-\theta})^{n-1-i} - 1 \right| < \frac{1}{n} \qquad \forall i, \tag{6.2}$$

which is equivalent to

$$\left(\frac{1}{1-\theta}\right)^{n-1} < 1 + \frac{1}{n} ,$$

that is

$$\theta < 1 - \frac{1}{(1+\frac{1}{n})^{\frac{1}{n-1}}} = \frac{\varepsilon_n}{1+\varepsilon_n} . \tag{6.3}$$

It follows from

$$F_h F_{h-1} \Delta > 1 + \frac{1}{\varepsilon_n}$$

that

$$q_h q_{h-1} \Delta = \frac{\Delta}{|b-a|} = \frac{1}{\theta} > 1 + \frac{1}{\varepsilon_n} ,$$

hence (6.3) holds and (6.2) is satisfied. Lemma 5.1 may be applied to conclude the proof. □