8.1 The case of simple roots

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THEOREM 7.5 (Chen's Main Theorem). *Let $f(x)$ be an integral polynomial of degree $n \geq 3$ with at least 3 variations. Let h be the smallest positive integer for which*

$$F_{h-1}^2 \Delta > 1,$$

and let m be the smallest positive integer such that

$$m > \tfrac{1}{2} \log_\phi n .$$

Let $k = h + m$. For an arbitrary continued fraction $\gamma = [c_0, c_1, \ldots] > 0$, consider the polynomial f_{k+1} constructed by F_k. If V is the number of variations of f_{k+1} then the polynomial f has a unique positive root in $\left(\frac{p_{k-1}}{q_{k-1}}, \frac{p_k}{q_k}\right)$ and V is its multiplicity.

Proof. After h steps, the polynomial f_{h+1} might have no variations, and then f_{k+1} will have no variations. If f_{h+1} has V variations, by Chen's Theorem 1 it has a positive root in the right half plane. The partial quotients c_i are ≥ 1 for $i > h$, and so we may apply Chen's Theorem 2. □

8. A NEW PROOF OF VINCENT'S THEOREM

In this section we give a new and simpler proof of Vincent's theorem, which in turn improves on Chen's result. For the sake of clarity, we prefer to deal separately with the two cases of simple and multiple roots.

8.1 THE CASE OF SIMPLE ROOTS

In the case of simple roots, we show that Vincent's theorem holds under the only assumption

$$\Delta F_h F_{h-1} > \frac{2}{\sqrt{3}}$$

independently of the polynomial degree n.

Our proof depends on the following result by Obreschkoff [30, p. 81].

LEMMA 8.1. *Let $f(x)$ be a real polynomial with V variations in the sequence of its coefficients; let V_1 be the number of variations of the polynomial $f_1(x) = (x^2 + 2\rho x \cos\varphi + \rho^2) f(x)$ (where $\rho > 0$ and $|\varphi| < \frac{\pi}{V+2}$).*

Then $V \geq V_1$, and the difference $V - V_1$ is an even number.

The analogous result, for the polynomial $(x+p)f(x)$, with $p \in \mathbf{R}^+$, can also be proved by a slight modification of Obreschkoff's proof. In the sequel, this extended version comprising both cases, will be referred to as Lemma 8.1.

For $V = 1$ we get the following

COROLLARY 8.2. *If a real polynomial has one positive simple root x_0, and all the other (possibly multiple) roots lie in the sector:*

$$S_{\sqrt{3}} = \{x = -\alpha + i\beta \mid \alpha > 0 \textit{ and } \beta^2 \leq 3\alpha^2\}$$

then the sequence of its coefficients has exactly one sign variation.

Obreschkoff argued by contradiction. We give here a simple constructive proof of the corollary.

Proof. It suffices to prove that if

$$f(x) = \sum_{k=0}^{n} a_k x^{n-k}$$

is a real polynomial whose sequence of coefficients has one variation, then both

$$f_\alpha(x) = (x+\alpha)f(x),$$
$$f_{\alpha,\beta}(x) = \left[(x+\alpha)^2 + \beta^2\right]f(x)$$

have exactly one variation. Starting with $f(x) = (x - x_0)$ and iterating the above argument for any root in $S_{\sqrt{3}}$ one gets the claim.

Indeed,

$$f_\alpha(x) = \sum_{k=0}^{n+1} b_k x^{n+1-k}$$

where (setting $a_{-1} = a_{n+1} = 0$)

$$b_k = a_k + \alpha a_{k-1} \quad \text{for } k = 0, 1, \ldots, n+1.$$

If f has one variation, then there exist indices j and i, with $0 \leq j < i \leq n$, such that

$$\begin{aligned} &a_{-1}, a_0, \ldots, a_{j-1} \geq 0 \quad \text{and} \quad a_j > 0, \\ (8.1) \qquad &a_{j+1} = \cdots = a_{i-1} = 0 \quad (\text{if } i > j+1), \\ &a_i < 0 \quad \text{and} \quad a_{i+1}, \ldots, a_{n+1} \leq 0, \end{aligned}$$

whence

$$b_0, \ldots, b_{j-1} \geq 0 \quad \text{and} \quad b_j > 0,$$
$$b_{i+1} < 0 \quad \text{and} \quad b_{i+2}, \ldots, b_{n+1} \leq 0.$$

If $i = j + 1$ only the sign of b_i is unpredictable; if $i \geq j + 2$, then $b_{j+1} > 0$ and $b_i < 0$ (and $b_{j+2} = \cdots = b_{i-1} = 0$ if $i > j + 2$). In any case, the polynomial f_α has just one variation.

Now consider

$$f_{\alpha,\beta}(x) = \sum_{k=0}^{n+2} d_k\, x^{n+2-k} \quad \text{where} \quad d_k = a_k + 2\,\alpha\, a_{k-1} + (\alpha^2 + \beta^2)\, a_{k-2}$$

with

$$a_{-2} = a_{-1} = 0 = a_{n+1} = a_{n+2}\,.$$

If (8.1) still holds (including $a_{-2} = a_{n+2} = 0$), then

$$d_0, \ldots, d_{j-1} \geq 0 \text{ and } d_j > 0 \ \ldots \text{ and } d_{i+2} < 0 \text{ and } d_{i+3}, \ldots, d_{n+2} \leq 0.$$

If $i \geq j + 3$ or $i = j + 2$, then the sequence d_k has one variation.

If $i = j + 1$ we show that

$$d_{j+1} < 0 \quad \text{implies} \quad d_{j+2} \leq 0$$

and this suffices to prove that the sequence d_k has only one variation.

The inequality

$$d_{j+1} = a_{j+1} + 2\,\alpha\, a_j + (\alpha^2 + \beta^2) a_{j-1} < 0$$

with $a_{j-1} \geq 0$, implies $a_{j+1} + 2\alpha a_j < 0$. Therefore

$$\begin{aligned} d_{j+2} &= a_{j+2} + 2\,\alpha\, a_{j+1} + (\alpha^2 + \beta^2)\, a_j \\ &\leq 2\,\alpha\, a_{j+1} + (\alpha^2 + 3\alpha^2)\, a_j = 2\,\alpha(a_{j+1} + 2\,\alpha\, a_j) \leq 0 \end{aligned}$$

(since $a_{j+2} \leq 0$) and this completes the proof. □

Now we prove the theorem in our stronger form.

Proof. We use the same notation as before. Suppose h is such that

$$\Delta F_h F_{h-1} > \frac{2}{\sqrt{3}}\,.$$

Since $\dfrac{2}{\sqrt{3}} > 1$, we know by the previous argument that all the quadratic irreducible factors of the transformed polynomial $\phi(x)$ have nonnegative coefficients and that at most one linear factor of $\phi(x)$ has coefficients with opposite signs. This happens if and only if there is a positive root x_0 of $f(x)$, which belongs to the interval (a, b).

We only need to show that all the roots of $f(x)$ different from x_0 are mapped into the sector $S_{\sqrt{3}}$. In this case the transformed polynomial $\phi(x)$ has exactly one variation by Corollary 8.2.

Once again we consider the map $\mathcal{F}\colon \mathbf{C} \to \mathbf{C}$ defined by

$$y = \mathcal{F}(x) = \frac{x-a}{b-x}. \tag{8.2}$$

In the previous section we observed that the map (8.2) transforms the circle

$$\left| x - \frac{a+b}{2} \right| = \frac{1}{2}|b-a| \tag{8.3}$$

into the line $\operatorname{Re}(y) = 0$, and the exterior of this circle into the half-plane $\operatorname{Re}(y) < 0$ (see Fig. 1a).

But another property of (8.2) is relevant: the circles passing through the points a, b are sent into lines through the origin of the complex plane. More precisely, the lines

$$\operatorname{Im}(y) = \pm s \operatorname{Re}(y) \quad (s \in \mathbf{R}^+)$$

are the images of the circles centered at

$$c^{\pm} = \frac{a+b}{2} \pm i\frac{|b-a|}{2s},$$

with radius

$$r = \frac{|b-a|}{2}\sqrt{1 + \frac{1}{s^2}}.$$

It easily follows (see Fig. 2) that the sector S of the complex plane defined by

$$\operatorname{Re}(y) < 0 \quad \text{and} \quad |\operatorname{Im}(y)| \leq |s| \cdot |\operatorname{Re}(y)| \tag{8.4}$$

is the image under $\mathcal{F}$ of the exterior of the eight-shaped figure R given by the union of the two disks

$$\left| x - c^{\pm} \right| \leq r.$$

A point x at distance greater than $2r$ from a point of the segment with endpoints a, b cannot be in the interior of R, and hence $\mathcal{F}(x) \in S$.

To ensure the existence of at most one variation in the transformed polynomial, we must require that $s = \sqrt{3}$, and hence consider the particular sector $S_{\sqrt{3}}$.

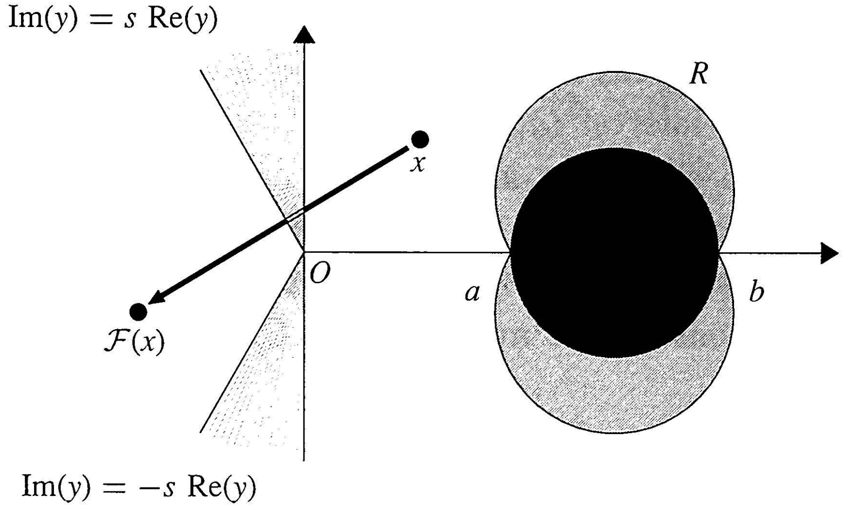

FIGURE 2

Now $r = \dfrac{|b-a|}{\sqrt{3}}$ and $\Delta F_h F_{h-1} > \dfrac{2}{\sqrt{3}}$ imply

$$\Delta q_h q_{h-1} > \frac{2}{\sqrt{3}},$$

that is

$$\frac{\Delta}{|b-a|} > \frac{2}{\sqrt{3}} \quad \text{or} \quad \Delta > 2r.$$

Only the root x_0 is in the interior of R, and so any other root is mapped by (8.2) into S. □

REMARK 8. To compute the continued fraction expansions of algebraic numbers which occur as zeros of integer polynomials, an interesting class of polynomials is given by the *reduced polynomials* (see [12] and [14]). A polynomial $f(x)$ is *reduced* if it has a unique root $x_0 > 1$ and all its other roots x_j satisfy $|x_j| < 1$ and $\mathrm{Re}(x_j) < 0$. A reduced polynomial does not necessarily have a unique variation, nor is a polynomial with a unique variation necessarily reduced. But it is interesting to observe that the machinery (3.1) of Vincent's theorem establishes a deep connection between the two classes of polynomials. In [12], the authors give a brilliant proof that, for sufficiently large h, the polynomial f_h is reduced. A remarkable difference between reduced polynomials and polynomials with a single variation is that we can immediately check that a polynomial has a single variation, while it is not so immediate to verify that a polynomial is reduced. A possible test is given by the combined use of Theorems 40.2 and 42.1 of [29]. Since we have replaced the transformation

$$x \leftarrow \frac{p_{h-1} + p_h x}{q_{h-1} + q_h x}$$

by

$$x \leftarrow \frac{a + bx}{1 + x},$$

we have to replace the unitary circle by the circle of radius $\frac{q_h}{q_{h-1}}$. We obtain a reduced polynomial if we require that under the map (8.2) the image $\mathcal{F}(x_j) = \frac{x_j - a}{b - x_j}$ of a root x_j of $f(x)$, different from x_0, is such that

$$|\mathcal{F}(x_j)| < \frac{q_h}{q_{h-1}}, \quad \text{and} \quad \operatorname{Re} \mathcal{F}(x_j) < 0. \tag{8.5}$$

Let $t = \frac{q_h}{q_{h-1}}$, and consider the Apollonius circle

$$\left| \frac{x - a}{x - b} \right| = t. \tag{8.6}$$

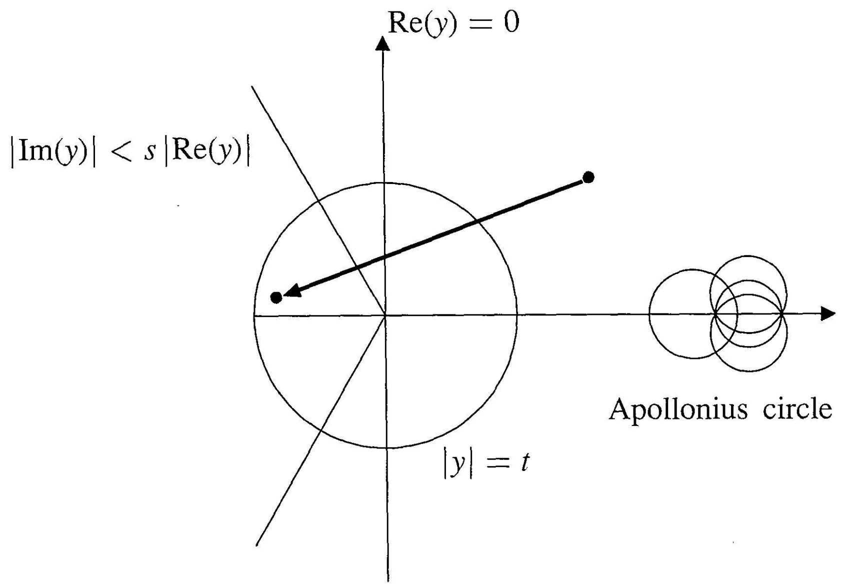

FIGURE 3

$\mathcal{F}$ maps the exterior of the circle (8.6) into the interior of the circle $|y| = t$. Hence the first condition in (8.5) means that x_j must be outside the circle (8.6). The diameter of the circle (8.6) lies on the real axis, and its endpoints are

$$u = \frac{a - tb}{1 - t}, \quad v = \frac{a + tb}{1 + t}.$$

Clearly $v > 0$; moreover

$$u = \frac{\dfrac{p_{h-1}}{q_{h-1}} - \dfrac{q_h}{q_{h-1}}\dfrac{p_h}{q_h}}{1 - \dfrac{q_h}{q_{h-1}}} = \frac{p_{h-1} - p_h}{q_{h-1} - q_h} > 0 \,.$$

It follows that this circle is entirely contained in the right half plane. A root x_j different from x_0 (see Fig. 3) lies outside the circle (8.3) if h is large enough to have $\Delta > |b-a|$. It follows that $\operatorname{Re}\mathcal{F}(x_j) < 0$, and hence it is external to the circle (8.6) corresponding to the value $h+1$. Hence the condition

$$F_{h-1}F_{h-2}\Delta > 1$$

ensures that the polynomial f_{h+1} is reduced.

8.2 THE CASE OF MULTIPLE ROOTS

Obreschkoff's Lemma 8.1 yields the following

COROLLARY 8.3. *Let $f(x) = (x-x_1)(x-x_2)\cdot\ldots\cdot(x-x_r)$, where $x_i \in \mathbf{R}^+$. Then*

$$f_1(x) = (x^2 + 2\rho x\cos\varphi + \rho^2)f(x)\,, \quad \rho > 0\,, \quad |\varphi| < \frac{\pi}{r+2}$$

has exactly r variations. More generally, a polynomial having r positive real roots and all its other roots in the sector

$$S = \left\{x \,\middle|\, x = -\rho(\cos\varphi + i\sin\varphi), \quad \rho > 0, \quad |\varphi| < \frac{\pi}{r+2}\right\}$$

has exactly r variations.

This allows us to extend Vincent's theorem to the case of multiple roots. Suppose the polynomial $f(x)$ has multiple roots, and let Δ be their least distance. If h is sufficiently large to verify

$$F_h F_{h-1}\Delta > 1\,,$$

at most one root x_0 lies in (a,b), but since this root may have multiplicity r, f_h has 0 or at least r variations. It will have exactly r variations if we can ensure that $x_0 \in (a,b)$ and that the other transformed roots lie in the sector

$$S = \{y \mid \operatorname{Re} y < 0, \quad |\operatorname{Im} y| < |\tan\varphi| \cdot |\operatorname{Re} x|\}\,,$$

where $\varphi = \dfrac{\pi}{r+2}$. Let $s = \tan\dfrac{\pi}{r+2}$ and let us make the appropriate substitutions into (8.4). We have proved